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Stochastic Essentials for the Risk Management of Credit Portfolios

By Ludger Overbeck, Frankfurt/M.,
and Gerhard Stahl, Bonn*

I. Introduction

Credit risk models play an increasingly important role in banks' risk management strategy. The main economic reasons for this are: the globalization of markets brings new opportunities with more complex risks, e.g. emerging markets, the need for optimal capital allocation through advanced risk measurement, the use of derivatives on a significant scale that creates new forms of default risk, and the outstanding importance of credit risk to the prospect of all types of financial institutions.

The calculation of potential losses and of the required capital cushion becomes an important factor in competition against the background of increasingly scarce capital resources. Furthermore, risk-adjusted return must be precisely measured (RAROC = Risk Adjusted Return on Capital, see Matten, (12)) in the context of cash flow, capital management, and shareholder value.

VaR models depict market price risks in aggregated form, i.e. portfolio value changes depending on market price changes in basic variables (risk factors) in a single risk figure (value at risk). From a theoretical point of view the model structure underlying VaR models is elementary, being defined by data-driven statistical time series models; there is no substantial economic model providing support. ARCH models are a good example of the statistical modelling of stylized facts of financial time series (volatility clusters, leptokurtosis). On the other hand the functional relationship between the risk factors is known in advance. Hence, the main role of statistical modeling is the reduction of the huge number of input dimension of a VaR model. The term mapping is com-

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monly used to denote all kinds of dimension reducing techniques applied in risk management (e.g. beta factor models and principal component analysis).

In contrast to VaR models for market risks, credit risk models focus in general on analysis of the effects of creditworthiness changes on changes in the value of a credit portfolio. The portfolio view has special importance here, especially in the determination of main concentration risks and for correlated defaults. The additional complexity implies that, besides the aforementioned statistical models, the full range of econometric and traditional methods of multi-variate statistics will come into play.

The following diagram from (2) highlights the basic scheme of a factor model to estimate the correlation of default events: It emphasizes – compared to market risk models – the complex hierarchy of credit risk models. The firm's risk is decomposed into systematic and specific risk. The systematic risk is again decomposed into country and industry risks which again are driven by some global factors. The interplay of economic and statistical modeling is obvious from the considered variables.

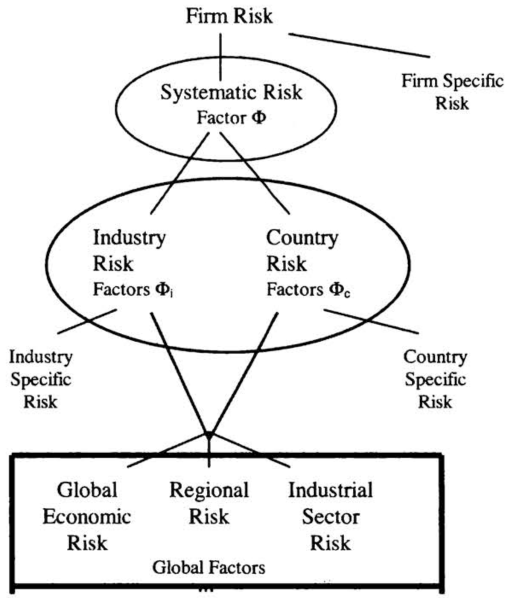

Figure 1

This paper introduces basic methods in credit risk management, especially from a portfolio view. The paper is organized as follows. The second section looks at an extreme, but the most important, case of change in a counterparty's creditworthiness – namely his default. In the third part of the paper a method is introduced for modelling the default using techniques from option price theory, Merton (13) Black and Scholes (5). Our presentation is also based substantially on Kealhofer (10) and Vasicek (15). The fourth section deals with dependence models, which form the basis for modeling portfolio effects. Default correlations and joint default probabilities are calculated consistently within the model. The fifth section sketches how the distribution of losses might be determined.The resulting loss distribution is also basis of the new Basle Proposal on the revision of the 88-Accord. The sixth section informs on risk figures for determining required capital resources. The final section on risk allocation presents elementary methods of risk management.

II. Credit Portfolios

Let us consider a portfolio of transactions with m counterparties. The time horizon at which the loss distribution is to be determined is fixed, namely 1 year. The random variable portfolio loss can then be written as

$$L = \sum_{i=1}^{m} L_i, \tag{1}$$

where L_i is the loss associated with transaction i. In the simplest model (pure default mode)

$$L_i = w_i \mathbf{1}_{D_i}, \tag{2}$$

where D_i is the default event and w_i is the exposure amount assumed to be known with certainty. The default event itself is at that time the realization of a Bernoulli random variable. i.e. the indicator $\mathbf{1}_D$. The event default is coded by D and we use the short hand notation $P(D)$ to represent the probability of default.

More elaborate models (like CreditMetrics (6)) assume multinomial variables in order to account for all rating categories, e.g. $\{AAA, \dots, D\}$. The random variable loss associated with counterparty i takes then the form

(3)
$$L_i = \sum_{r=AAA}^{D} l_{r,i} \mathbf{1}_{D_{r,i}},$$

where $D_{r,i}$ denotes the event that counterparty i is in rating class r and $l_{r,i}$ is the loss, or more precisely the change in value, associated with the migration of i to rating r. The loss amount is usually deterministic given the total amount of exposure and the given migration, i.e. $l_{r,i}$ is a function of r, the exposure and the present rating of i.

The full asset value model (13), e.g. implemented by KMV (11), assumes

(4)
$$L_i = L(i, A_1(i)),$$

where $(A_t(i))_{t\geq 0}$ is the stochastic process governing the asset value process of counterparty i. In the binary model ("default only") we set

(5)
$$L(i, A_1(i)) = w_i \mathbf{1}_{\{A_1(i)<C_i\}},$$

where C_i is the default boundary. We will basically consider the last approach. This approach is implemented in our Monte-Carlo-Simulation study and is also used for large loan portfolios of commercial banks. But similar results also hold for more general models as described in equation (1).

III. Default Models

The asset value model of Merton (13) and Kealhofer (10) is based on an option price theory approach which assumes that a companys default probability is determined by the dynamics of its asset value. By analogy with classical applications of option price theory, this approach supplies credit valuation methods which are independent of subjective assessments and intentions of the company. A default occurs when a company's value is lower than the amount to pay at credit maturity. This probability can be calculated if net asset value follows a stochastic process. By analogy with the Black-Scholes world, the processes on which the default depends are modelled by diffusions.

Diffusion Models – Merton's Asset Value Model

The simplest model works for listed companies. It states that a company defaults when its share value falls below a certain size. This ignores, of course, the fact that share value is only one indicator of actual asset value or net worth. The seminal papers by Black & Scholes (5) and Merton (13) therefore proposed viewing the share as an option on asset value. This strategy was also pursued by Kealhofer (10). Our intention in the following paragraph is to derive the time dynamics of asset value, which we regard as the driving stochastic force A_t, from the dynamic development of share prices and several static information sources, e.g. balance sheet, liabilities, etc.

Modelling the Dynamics of Asset Value

The calculation of asset values is based on option price theory. The following stochastic differential equations are assumed to represent dynamic behavior of share prices $(E_t)_{t>0}$, more precisely the total value of the shares, and asset values $(A_t)_{t>0}$:

$$\begin{aligned} dA_t &= \mu_A A_t dt + \sigma_A A_t dW_t^A \\ dE_t &= \mu_E E_t dt + \sigma_E E_t dW_t^E. \end{aligned}$$

Here W^A and W^E denote two independent standard Brownian motions.

In order to motivate this framework, assume that all shareholders decide at time $T = 1$ to sell the company. They must first settle all debts of amount C. Their profit is therefore equal to

$$\max\{0, A_1 - C\},$$

which corresponds to the pay-off of a European call option with strike C and maturity $T = 1$. (It could also be assumed that shareholders can exercise their option at any time up to time T. In case of call options, however, this additional possibility leads to the same price (price American call = price European call), see, for example, Hull, (8)). This simplistic approach should be refined in an implementation. In general, though, the result is a function which describes share value as a function of asset value. This function must satisfy the conditions of the implicit function theorem, so that it can be solved locally in accordance with the variables. In particular, this is intended to guarantee invertibility with regard to the variable "asset value". It follows from option price theory, e.g. the Black-Scholes formula, that with, for example,

$$\begin{aligned} E_t &= V(T, C, \sigma_A, A_t) =: E(A_t, t) \\ A_t &= V^{-1}(T, C, \sigma_A, E_t), \end{aligned}$$

where V denotes a call option price. So only σ_A remains to be determined.

We apply the Itô formula to $E(A, t)$ yielding:

$$\sigma_E E_t dW_t^E = E_a(A_t, t)\sigma_A A_t dW_t^A,$$

where E_a denotes the derivative of E with respect to a. It follows that

$$\frac{\sigma_E}{\sigma_A} = \frac{A_t E_a(A_t, t)}{E_t}.$$

This gives σ_A, and hence also A_t, and a time series for the asset values. For a given time series of asset values, we give below two possible ways of defining the default event at planning time $t = 1$ or for a planning period.

Classical Approach

Only asset value at planning time $t = 1$ determines default. The set of all paths leading to the default is then

$$D = \{\omega | A_1(\omega) < C\}.$$

This leads to simple formulae for default probability as a function of σ_A, mean return μ_A, initial value A_0, and default point C:

$$P(D) = \Phi(\sigma_A^{-1}(\ln(C/A_0) - (\mu_A - \frac{1}{2}\sigma_A^2)).$$

(Φ denotes here and below the distribution function of the standard normal distribution. The default point C may naively thought to be the discounted future obligations or only the debt due at time 1. In practice, it is a calibrated function depending on the term structure of debts.)

Definition with First Passage Times

Assuming a company defaults when, at some time to planning time, liabilities can no longer be served, the paths leading to the company's bankruptcy must be defined using first passage times:

$$D = \{\omega | A_t(\omega) < C, \text{ for some } t \leq 1\}.$$

For geometric Brownian motions, first passage times have been studied in detail in stochastic process literature. Detailed application to credit risk is found, for example, in Zhou (17), where default probabilities are also given.

Local Times and First Passage Probabilities

A third possible way to determine the default event would be to equate default with asset value falling below value C for a stated period, e.g. two days. The mathematical tool to be used here would be the local time, or the occupation density, of stochastic processes. These have also been studied in detail, cf. Revuz/Yor (26).

IV. Correlation Model

As mentioned before the main focus of this paper is on modeling correlations. In that respect the diffusion models introduced in section 3 offer a distinctive advantage by enabling correlation to be modeled consistently. We now assume a portfolio with m counterparties, each of which has exposure w_i, $i = 1, \cdots, m$. For simplification purposes, we assume that w_i are non-random. Relevant examples of random exposures are traded products like swaps, options etc. and also traditional lending products like the utilization of credit lines.

Stochastic Model

It is assumed that the dynamic behavior of the portfolio's driving risk factors, namely the vector of all asset value processes, is represented by the following system of stochastic differential equations

$$dA_t(i) = \mu_i A_t(i)dt + \sigma_i A_t(i)dZ_t(i), \quad i = 1, \ldots, m \tag{6}$$

with μ_i denoting expectation and σ_i asset volatilities. The process $Z_t = (Z_t(1), \cdots, Z_t(m))$ follows a multi-dimensional Brownian motion with (asset) correlation matrix $(\rho_{ij})_{i,j=1,\ldots,m}$. The credit event "default", D_i, is just the event in which the *i–th* firms asset value falls under its default point C_i – either at the end of the planning period $T = 1$ or at some time in the time interval $[0, T]$.

Pairwise Defaults

Assume default is defined by the event "asset value at time 1 is less than C_i", the joint default probability derives from

$$P(A_1(i) < C_i, A_1(j) < C_j).$$

The evaluation of the integral of the subset $\{(x,y) \mid x < C_i \wedge y < C_j\}$ w.r.t. a bivariate Gaussian random vector with asset correlation (ρ_{ij}), marginal standard deviations σ_i and σ_j yields the desired probability.

For the portfolio view, r_{ij}, the default correlations, are the important values in order to calculate portfolio risk. They appear in the formula for the portfolio risk, i.e. the standard deviation σ_p of L as defined in (1) by

$$\sigma_p^2 = Var(L_p) = \sum_{i,j=1}^{m} w_i w_j \sigma_{D_i} \sigma_{D_j} r_{ij},$$

where the standard deviation of single exposure σ_{D_i} is equal to $\sqrt{P(D_i)(1 - P(D_i))}$.

Default Correlations

The correlation r_{ij} are correlations of the variables $\mathbf{1}_{D_i}$ and $\mathbf{1}_{D_j}$. They are derived uniquely by the formula for the correlation of Bernoulli distributed random variables from the joint default probabilities $P(D_i \cap D_j)$, the specific default probabilities, and the asset correlations,

$$r_{ij} = \frac{P(D_i \cap D_j) - P(D_i)P(D_j)}{\sigma_{D_i}\sigma_{D_j}}.$$

The default correlation depends functionally on the joint default probability and the specific default probabilities. The joint default probabilities are again functions of the single default probabilities and the asset correlations. For this reason, not only the asset correlation is important for the default correlation, but also the single default probabilities, as the following table shows.

Table 1

firm	P2	ASCORR	CORR
2	0.0002	0.476969601	0.038052303
3	0.0003	0.65	0.087200222
4	0.0006	0.476969601	0.054738525
5	0.0009	0.476969601	0.062082239
6	0.0009	0.522494019	0.076073885
7	0.001	0.476969601	0.064085023
8	0.0014	0.476969601	0.070758029
9	0.0016	0.476969601	0.07351272
10	0.0016	0.65	0.149195309
11	0.0017	0.522494019	0.091446648
12	0.0017	0.614532343	0.132810343
13	0.0021	0.564933624	0.115560309
14	0.0026	0.476969601	0.083980448
15	0.0026	0.550363516	0.11507797
16	0.0026	0.606547607	0.143973763
17	0.0042	0.476969601	0.094865018
18	0.015	0.550363516	0.164429433
19	0.0205	0.606547607	0.208652518

Table of correlations. This example is based on a portfolio of 19 firms. The second column in the table contains the associated default probabilities, $P2$, of the firms 2 to 19. The default probability of firm 1 equals 113bp. The figures in the third column, ASCORR, are the asset correlations between firm 1 and firm $j, j = 2, \cdots, 19$. The last column, CORR, contains the default correlations.

Counterparties with lower default probability have lower default correlations, with the same asset correlation, than those with high default probability.

Implementation

In the application of the presently described model to bankwide RAROC-calculations, default probabilities themselves are in practice not based on diffusion models. They are taken over from the calibration of historical default frequencies in rating classes, either based on internal or external ratings. Default probabilities and data on the structure of liabilities and a company's present assets then, however, determine the so-called "standardized distance to default".

Remark on First Passage Time

If we use first passage times to model a default event, then

$$P(D_i \cap D_j) = P(\ \exists s, t \leq 1,\ : A_t(i) < C_i, A_s(j) < C_j).$$

This probability is stated explicitly in Zhou (17).

The interesting question now is: how, with given asset correlations and default probabilities, the correlations in the two presented models behave. It is obvious that, with the same standard deviations of asset values and the same asset correlation, the model with first passage times produces a lower joint default probability.

Models for Asset Correlation

We outline two approaches to calculate asset correlations.

Factor Model

One widespread procedure to determine the desired correlation is the application of factor analysis to asset returns. In that set-up any asset return is decomposed in a systematic portion and an unsystematic portion:

$$\ln A_t(i) = \beta_i F_t(i) + \epsilon_t(i), i = 1, \ldots, m.$$

By definition the random variables ϵ_t – associated with the unsystematic portion – are independent of each other and of the factors. The factor $F(i)$ models the incorporation of the *i–th* counterparty into the factor model. The correlations between two asset returns equal to

$$\frac{\beta_i \beta_j \operatorname{corr}(F(i), F(j))}{\sigma_i \sigma_j}, \quad i, j = 1, \ldots, m, \; i \neq j.$$

The factor model may be implemented smoothly, for example, to a hierarchical structure, as indicated in Figure 1, shown in the introduction.

Implicit Covariances

Based on the option price framework it is possible to derive asset correlation implicitly from equity correlations. To that end the Itô formula is applied to the product $A_i(t, E_t(i))A_j(t, E_t(j))$, which gives an equation for the mixed second moments $E(A_t(i)A_t(j))$:

$$
\begin{aligned}
&d\ \mathbb{E}[A_t(i)A_t(j)]/dt = \\
&\mathbb{E}[\partial A_i(t,E_t(i))A_j(t,E_t(j))\mu_{i,E}E_t(i)] + \mathbb{E}[A_i(t,E_t(i))\partial A_j(t,E_t(j))\mu_{j,E}E_t(j)] \\
&\ + \mathbb{E}[\partial^2 A_i(t,E_t(i))A_j(t,E_t(j))\sigma_{i,E}E_t(i)] + \mathbb{E}[A_i(t,E_t(i))\partial^2 A_j(t,E_t(j))\sigma_{j,E}E_t(j)] \\
&\ + \mathbb{E}[\partial A_i(t,E_t(i))\partial A_j(t,E_t(j))\mathrm{cov}_{ij,E}E_t(j)E_t(i)]\ .
\end{aligned}
$$

Here, $cov_{ij,E}$ denotes the equity covariance. The desired asset correlation is recovered immediately from the formula for the mixed moment of the second order.

Joint Default Probabilities

Joint default probability is a much better indicator for the analysis of concentration in a portfolio. The joint default probability is a function of specific default probabilities and asset correlation. It is relatively simple to calculate default probability pairs in the Merton model if the covariance structure of asset returns is known.

Conditional Default Probabilities

In the context of portfolio analysis it is helpful to calculate scenarios where certain counterparties default. Besides the forecast for the loss from this single exposure, it is also important to estimate the impact for the entire portfolio. The decisive influence comes from the change in default probabilities. The original default probabilities must be replaced by conditional default probabilities, given that the considered counterparty, say firm 1, has defaulted. Clearly the new default probabilities $\tilde{P}$ are obtained by

$$\tilde{P}(D_j) = \frac{P(D_1 \cap D_j)}{P(D_1)}.$$

As a rule, this entails a very big increase of the likelihood of default, especially for counterparties with an originally low default probability, as can be seen from the following table. The underlying portfolio is the same as in Table 1.

This non-linear increase in default probability clearly has to do with correlation structure. The default of firm 1 implies that the influencing factors are in an unfavourable situation. Counterparties having high correlation with this defaulted counterparty depend on the same factors, or at least on highly correlated factors. This is why the market situation is

Table 2

firm	P2	JDP	CONDPRO	INCREASE
2	0.0002	6.43012E-05	0.004834675	24
3	0.0003	0.000176984	0.01330708	44
4	0.0006	0.000161532	0.012145298	20
5	0.0009	0.000225231	0.016934663	19
6	0.0009	0.000273294	0.02054844	23
7	0.001	0.000245337	0.018446408	18
8	0.0014	0.000321698	0.024187793	17
9	0.0016	0.000357864	0.026907049	17
10	0.0016	0.000704382	0.052961085	33
11	0.0017	0.00045417	0.034148119	20
12	0.0017	0.000649376	0.048825247	29
13	0.0021	0.000633941	0.047664709	23
14	0.0026	0.000524492	0.039435476	15
15	0.0026	0.000705904	0.053075461	20
16	0.0026	0.000874471	0.065749724	25
17	0.0042	0.000758666	0.057042523	14
18	0.015	0.002489111	0.187151197	12
19	0.0205	0.003659697	0.275165154	13

The second column $P2$ is identical with column 2 in Table 1 and denotes the default probabilities. The third column contains the joint default probabilities, JDP for short. The fourth column gives the conditional default probabilities, $\tilde{P}(D_j)$. The last column is just the ratio $\frac{CONDPRO}{P2}$.

also unfavourable for these counterparties and they receive a higher default probability. However, this theoretically determined default probability overestimates actual risk, for in reality the correlation structure of the counterparties has already changed before default. This fact is underpinned empirically by a study of the KMV cooperation. But it is also clear from the fact that business relations have usually been reduced before default.

V. Loss Distribution

Up to now, we considered only portfolio risk, i.e. the standard deviation and variance of the variable L. The whole distribution of this random variable is at the core of a portfolio view of credit risk. At first sight, L has an almost elementary form, but straight forward caculations do not work here because of numerical problems e.g. the determination of binomial coefficents in the form $\binom{1000000}{17}$.

Basic Formula

Let m be the number of counterparties in the portfolio and w_i the exposure of counterparty i, the first step is to calculate the probability of k counterparties defaulting:

$$P\left(\sum_{i=1}^{m} \mathbf{1}_{D_i} = k\right)$$

$$= \sum_{\{i_1,\ldots,i_k\}\subset\{1,\ldots,m\}} P(A_1(i_l) < C_{i_l}, l = 1,\ldots,k, A_1(j) > C_j, j \in \{1,\ldots,m\} \setminus \{i_1,\ldots,i_k\}).$$

The latter is the probability that an k-dimensional normal distribution is in its bottom left-hand corner. This value is very cumbersome to calculate, because it is of very small size. On the other hand, the number of possibilities of selecting these k default candidates is very large and almost incomputable.

1. Monte-Carlo Simulation

An approximation to the exact analytical formula of the loss distribution is given by the Monte Carlo simulation of possible losses. The simulation technique can be described briefly as follows: Create

- $\Phi_i(j)$ samples of the factors F_j in the factor model, in order to simulate correlation effects, and
- ϵ_i^j independent, normally distributed random variables with variances as realizations as samples of company-specific risk.

Then, for each $i = 1, \cdots, N$ (= number of simulations),

$$r_i^j = \beta_j \cdot \Phi_i(j) + \epsilon_i^j, j = 1,\ldots,m$$

is the vector of the simulated asset return.

The empirical distribution function

$$\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{(0,x)}\left(\sum_{j=1}^{m} w_j \mathbf{1}_{\{r_i^j < C_j\}}\right)$$

is then a natural approximation of the loss distribution.

2. Analytical Approximation for Uniform Portfolios

In a portfolio where all customers have the same size of exposure, the same default probability p and all pairs of asset correlations are equal to ρ, the probability of k counterparties defaulting is explicitly given. The asset value processes are determined by the stochastic differential equation

$$dA_t(i) = \mu_i A_t(i)dt + \sigma_i A_t(i)dZ_t(i). \tag{7}$$

The process $Z_t = (Z_t(1), \cdots, Z_t(m))$ is a multi-dimensional Brownian motion with zero drift and correlation matrix given as

$$E((Z_t(i) - Z_s(i))(Z_t(j) - Z_s(j))) = \delta_{ij} + (1 - \delta_{ij})\rho \cdot (t - s).$$

(δ_{ij} is the Kronecker symbol, which is equal to 1 if $i = j$ and otherwise 0.) On the basis of this specific correlation structure, the process Z can be defined by

$$Z_t(i) = \sqrt{\rho} \cdot B_t + \sqrt{1-\rho} \cdot W_t(i).$$

Here, vector (B, W) is a standard $m+1$ dimensional Brownian motion, i.e. all components are independent. B can be regarded as the common economic factor which explains the systematic risk. W models the specific risk of individual counterparties.

A solution to the stochastic differential equation is

$$A_T(i) = A_0(i) \exp\left\{(\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i\sqrt{\rho}B_T + \sigma_i\sqrt{1-\rho} \cdot W_T(i)\right\}.$$

$P(L = \frac{k}{m})$ equals:

$$\binom{m}{k} \int_{-\infty}^{\infty} \Phi\left(-\frac{1}{\sqrt{1-\rho}}(c + \sqrt{\rho}x)\right)^k \left(1 - \Phi\left(-\frac{1}{\sqrt{1-\rho}}(c + \sqrt{\rho}x)\right)\right)^{m-k} \Phi(dx).$$

Under the assumption that counterparties have the same default probability p, c – the so-called (standardized) distance to default – is given by $c = \Phi^{-1}(p)$.

Infinite Number of Counterparties

For large portfolios, it is possible to assume that the number of single exposures is infinite (law of large numbers). The result obtained is that, for m towards ∞, the limit distribution has density f_∞, cf. Finger (7)

$$f_\infty(x) = \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \exp\Bigg(-\frac{1}{2r} \cdot (\sqrt{1-\rho} \cdot \Phi^{-1}(s) \\ -\Phi^{-1}(p))^2 + \frac{1}{2}(\Phi^{-1}(s))^2\Bigg).$$

This density plays a crucial rule in the new Basle proposal, (4), page 36. The risk weight function for corporate and retail borrowers is based on the inverse function of f_∞. An example of the loss distribution and its density is given in the following chart:

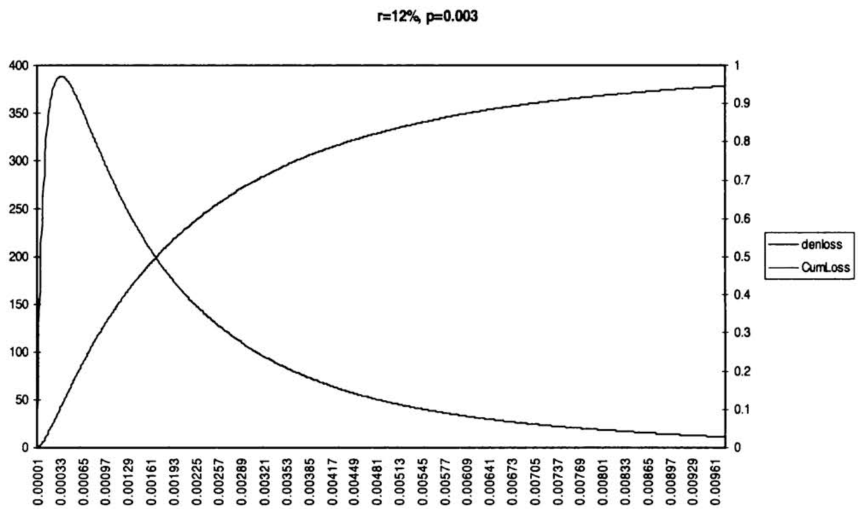

Figure 2

Figure 2 contains the density of losses for a uniform portfolio with infinite many counterparties and the associated cumulative distribution function. The calculation is done for a correlation of 12 % and a default probability of 0.3 %.

Of course, the assumptions that all counterparties have the same correlation and the same default probability does not really hold in practice. This leaves space for further investigations to modify this loss distribution, e.g. the possibility of admitting several different correlations is at hand.

VI. Economic Capital

In the context of credit portfolios, the economic capital serves as a cushion against extreme losses caused by the defaults of counterparties in the portfolio. Hence, the determination of economic capital related to business lines, portfolios or specific counterparties is of crucial importance. Risk measures currently used, like the VaR or the expected shortfall are parameters derived from the distrubtion of losses. This fact emphasizes the role of the loss distribution as an indispensible imput parameter. In the following we consider capital definitions based either on quantiles or the shortfall.

1. Quantiles

Since the mid 90 the VaR has become the most widely-used risk measure. According to internal risk management guidelines, a firm chooses a level of significance, α say, e.g. $\alpha = 0.9998$ and a particular holding period, say one year. Given these parameters, the α-quantile

$$P(L > EC(\alpha)) \leq 1 - \alpha,$$

is interpreted as the economic capital at confidence level α. This means that a bank that keeps an economic capital, $EC(99.98\%)$, will default only once in 5,000 years. The skewness of the distribution of L is also mirrored in the height of the capital cushion expressed as multiples of the portfolio standard deviation. A capital cushion determined by $EC(99.98\%)$ is pretty close 5% of the total exposure for the above portfolio. It is 12.5 times the portfolio deviation $\sigma_p \sim 0.004$. In addition to internal demands for calculating EC, external bodies, e.g. rating agencies or regulators use these figures. Rating agencies, especially use $EC(\alpha)$ as an important quantitative input in the rating process.

From a mathematical point of view it is not completely convincing to base risk measures on quantiles. Starting with the path-breaking paper by Artzner et al. (1), several studies revealed a number of methodological weaknesses in the concept of measuring risk by quantiles. By drawing up a catalogue of mathematical and material attributes which a risk measure should naturally fulfill, an axiomatic setting was derived. Risk measures which fulfill these axioms are called coherent. They are described in their basic features in the next subsection.

2. Coherent Risk Measures and Conditional Shortfall

A risk measure is coherent if and only if it is represented as a supremum of expectations under generalized scenarios $\mathcal{S}$:

$$\sup_{Q \in \mathcal{S}} E_Q(L), \tag{8}$$

where $\mathcal{S}$ is a family of probability measures on L's domain. A number of so-called downside risk measures, e.g. lower partial moments or mean excess functions are widespread in actuarial and financial applications. These moment-type risk measure have very appealing interpretations. In a recent paper by Jaschke and Küchler, see (9), it was shown that the shortfall risk above a quantile is close to a coherent risk measure, namely the worst conditional expectation. This motivates the usage of the shortfall risk as a definition for economic capital as the mean loss above a threshold u:

$$EC(u) = E(L|L > u). \tag{9}$$

In the following we consider the case of shortfall risk above a quantile, i.e., $EC(q_\alpha)$, where $q(\alpha)$ denotes the $\alpha-$quantile of a loss distribution. If L then had a normal distribution, the figure calculated in this way would not differ significantly from value-at-risk economic capital. For a $t-$distribution and the loss distribution f_∞ this difference is already large. Quantitative examples for the various definitions of economic capital under different distribution assumptions are given in Table 3, below.

VII. Contributory Capital

Consider the process of determinating economic capital EC and contributory economic capital CEC in more detail. It is an integrated two step process. First the overall EC is determined by means of calculating a loss distribution as indicated above. In a second step, the important questions, which are the contributions of business lines or even more detailed of each single transaction to the overall EC, is analysed. This type of marginal capital analysis is very important in credit risk management

Table 3

	Student(3)	N(0,1.73)	logNor(0,1)	N(1.64,2.16)	Weil(1,1)	N(1,1)	f(0.003,0.12)
STDV	1.73	1.73	2.16	2.16	1	1	0.0039
99%-QUANTILE	4.54	4.02	8.56	5.02	4.6	3.32	0.01922
ECNEW	6.99	4.61	13.57	5.76	5.6	3.66	0.0267401
INCREASE %	54	14	58	14	28	14	46

This table highlights the sensitivity of the determination of economic capital w.r.t. its definition and the choice of S. The scenarios are determined by just one distribution: the t(3), normal, log-normal, Weibull and f_∞. The first row contains the standard deviations, the second the VaR at 99%-level and the third gives EC(u), where u is the 99%-VaR. The last row is the ratio $\frac{VaR}{EC(VaR)}$. The parameters of the different distribution are obtained by moment matching for comparison. The Normal distribution with standard deviation has the same second moment as the student (3) distribution. The second moment of the standard lognormal coincides with the second moment of $N(1.64, 2.16)$ and the first two moment of Weil (1,1) and $N(1,1)$ are the same. The density $f(0.003, 0.12)$ is chosen as a benchmark against Student (3), i.e. the skewness of Student(3) and $f(0.003, 0.12)$ are similar which can also be seen from the INCREASE %-figure

since almost all bank businesses face credit risk. This is in contrast to market risk which is mainly concentrated to specific units of the bank (mainly treasury and trading areas). In the process of determining the contributory capital the overall EC is partitioned to each individual exposure. In the approaches we are going to describe the overall risk is the sum of all contributory capital. In effect this means that all transactions obtain the full diversification benefit. Each contributory economic capital figure might be viewed as the marginal capital of a single transaction that adds or contributes to the overall capital. We describe two approaches to determine these figures. The first approach is based on standard deviations within a well-known variance/covariance framework. The second approach is based on conditional expectations.

1. *Variance/Covariance Approach*

This approach is similar to Markowitz' theory on efficient portfolios. In this framework, volatility of the portfolio is used as the basic risk measure. For risk allocation purposes it is sufficient to measure the contribution of a single asset to the portfolio volatility. This setting is equivalent to the static Capital Asset Pricing Model if the market returns are identified with portfolio returns. Applying this approach to the problem of capital allocation for credit portfolios leads to the core question namely: calculating the amount a business unit – or most detailed an individual credit – that contributes to the total portfolio risk σ_p. To answer this question, we use the formula for volatility of a sum of random variables that splits up nicely the portfolio risk σ_p into risk contributions β_i in such a way that

$$\sum_{i=1}^{m} w_i \beta_i = \sigma_p.$$

The weighted sum of all risk contributions gives the risk in the total portfolio. It is clear from the definition of σ_p that

$$\sigma_p = \frac{1}{\sigma_p} \sum_{i=1}^{m} w_i \cdot \sum_{j=1}^{m} w_j \sigma_{D_i} \sigma_{D_j} \rho_{ij}.$$

Thus,

$$\beta_i = \frac{1}{\sigma_p} \sigma_{D_i} \sum_{j=1}^{m} w_j \sigma_{D_j} \rho_{ij}$$

is an intuitive figure that measures the risk contribution of credit i. This figure corresponds to the covariance of credit i, with total portfolio divided by portfolio volatility.

$$\beta_i = \frac{\operatorname{cov}(\mathbf{1}_{D_i}, \sum_{j=1}^{m} w_j \mathbf{1}_{D_j})}{\sigma_p} \tag{10}$$

The notation of β_i is obvious in analogy to beta-factor models used in market risk modeling. Furthermore, β_i is represented as the partial derivative of σ_p according to w_i, the weight of the i–th credit in the portfolio.

$$\beta_i = \frac{\partial \sigma_p}{\partial w_i} \tag{11}$$

In other words, an increase in the weight of this credit, by a small amount h in the portfolio, implies growth of σ_p by $h\beta_i$. The contribution of a business unit responsible for the loans 1 to k to portfolio volatility equals the sum of the individual contributions β_1 to β_k.

Assume, that the risk/return profile of the portfolio is defined in terms of standard deviations and risk contributions. In that framework the representation (11) may be used in the allocation[1] process. This active

[1] "Allocation" is explained in more detailed in following subsection on "Shortfall Contribution", property 2.

portfolio mangement leads to an improvement of the portfolio risk/return profile.

Capital Multiplier

Assume the definition and the amount of EC is agreed. As mentioned before it is yet often necessary to allocate the economic capital throughout the portfolio. This may be achvied by the application of so-called capital multipliers. A capital multiplier is defined by

$$\lambda = \frac{EC}{\sigma_p}.$$

Hence, the capital requirement for credit i is determined by

$$\delta_i = \lambda \beta_i.$$

The quantity δ_i is called analytic capital contribution of transaction i to the portfolio capital. For a unit in charge for credits 1 to $l(< m)$, the capital requirement is

$$\lambda \cdot \sum_{j=1}^{l} \sigma_{D_j}.$$

This multiplier is an auxiliary figure depending on the particular portfolio. This is due to the fact that, in contrast to the normal distribution, the quantiles of the loss distribution depend not only on standard deviation, but also on other factors such as correlations, default probabilities and weights. It is therefore unrealistic, after changing the portfolio, to obtain the same λ, even with the same σ_p, and thus the same economic capital.

2. *Conditional Expectations*

It is questionable to allocate economic capital efficiently, which, for example, is the 99.98% quantile of loss distribution, in terms of the β_i's. The β_i may yield a good partition of standard deviation, but not for a quantile in the tails of a skewed distribution. Assume a framework for risk allocation that is based on partial derivatives. This approach neglects in full the dependence of the quantile on correlations. For example, in this setting it is assumed that

$$\frac{\partial q_\alpha}{\partial \sigma_p}(0.9998) = \lambda.$$

One consequence of this very rough allocation process is that there may exist exposures for which more then 100% of its total exposure is required as its contributory economic capital. This effect is actually not desired.

A possible solution to circumvent this undesirable feature is the definition of the contributory capital of the i–th exposure as the mean loss of the i–th exposure in cases where the quantile of total portfolio is exceeded:

$$\gamma_i(\alpha) = E(L_i | L > q_\alpha),$$

or more generally for a large u

$$\gamma_i(u) = E(L_i | L > u).$$

This quantity is called shortfall contribution and it is consistent with the procedure in subsection VI.2. If overall capital is defined in terms of expected shortfall, the shortfall contribution measures the average loss of exposure i in those cases for which the capital is originally defined for. Consider for example a loan A which contributes in average more to the large losses over the threshold u. Hence, for this loan more capital should be required than for those which do not contribute to extreme portfolio losses.

First we list some formal properties of the capital allocation regime based on the shortfall contributions $\gamma_i(\cdot)$. The simulation study given in the next subsection exemplifies the difference in applying different methods to determine the risk contribution of counterparties.

Property 1 decribes a desired additivity property, which is interpreted that there is no waste of capital. The portfolio capital is distributed efficiently.

Property 2 relates the shortfall contributions to one-sided partial derivatives according to the weight w_i. This analytic property allows to use the shortfall contributions for capital allocation in order to improve the risk/return profile of the entire porfolio.

Finally *property 3* shows that the shortfall contributions constitute a coherent risk measure.

3. Formal Properties of Shortfall Contributions

1. The weighted sum of the shortfall contributions equals the expected shortfall of the portfolio:

$$\sum_{j=1}^{m} w_j \gamma_j(u) = EC(u). \tag{12}$$

2. Under some technical condition[2] we have that

$$\gamma_i(u) = \frac{\partial EC(u)}{\partial^+ w_i}.$$

3. γ_i is based on the (generalized) scenario, that large losses occur. Therefore, as detailed in (14) it is a coherent risk measure in the sense of (1).

Remarks:

- *Property 2* indicates that the proposed capital allocation $\gamma_i(u)$ may be used as a performance measure and an optimal allocation measure, as for example pointed out in Theorem 4.4 in Tasche (3). Optimality is meant in the following sense. First the notion of RAROC (Risk Adjusted Return on Economic Capital) is introduced.

  If r_i denotes the risk adjusted return of a loan and CEC_i the contributory capital of loan i, then $\mathrm{RAROC}_i = r_i/\mathrm{CEC}_i$. Let us assume loan 1 has an RAROC_1 larger than the overall portfolio RAROC defined by r_p/EC where r_p is the risk adjusted return of the portfolio.

  If one increases the exposure to counterparty 1 by a small amount then portfolio RAROC will be improved. In general it is only known that there is a positive increase of the exposure which leads to an improvement of the portfolio, but the actual maximal size of an increase of the exposure such that there is still an improvement of the portfolio RAROC is not known. But the same is true for optimal allocation with respect to the standard deviation of the portfolio, as in classical Markowitz theory.

[2] It is necessary that u cannot be represented as a sum of exposure weights w_i. However it is often claimed that the property holds more generally.

- $\gamma_i(u) < 1$. By construction the capital is always less then the exposure. A feature which is not shared by the risk contributions defined in terms of covariances.

- This shortfall contribution is a simple first statistic of the distribution of L_i given $L > K$. Other statistics like variance could be useful. (Conditional variance is probably not coherent.)

- The definition reflects a causality relationship. If counterparty i adds, in bad situations for the bank, more to the overall loss than counterparty j, then, as a consequence business with i should be more costly.

4. *Simulation Studies*

In the simulation studies we want to compare the two different allocation techniques outlined in this paper. We first applied them on two different portfolios. The third empirical example considers the case of allocating capital to business units. There are at least two reasons for the third example. First it might not be reasonable to allocate economic capital which is based on extreme risk to a single transaction, since the risk in a single transaction might be driven by short term volatility and not by the long term view of extreme risk. Yet, of course, on an aggregated level the long term view is without doubt the appropriate one. The second reason is more related to the computational feasibility of expected shortfall for each single transaction. In the binary world of defaults millions of simulations are necessary in order to obtain a positive contribution in extreme events for all counterparties.

The basic result of the simulations given below, is that analytic contributions produce a steeper gradient between risky and less riskier loans than tail risk contribution. A consequence of this steep gradient might be that the contributory capital is larger than 100%, as already mentioned before. In particular, loans with a high default probability but moderate exposure concentration require more capital in the analytic contribution method, whereas loans with high concentration require relatively more capital in the shortfall contribution method.

Portfolio A

The first simulation study is based on credit portfolio considered in detail in Overbeck, see (14). The particular portfolio consists of 40 coun-

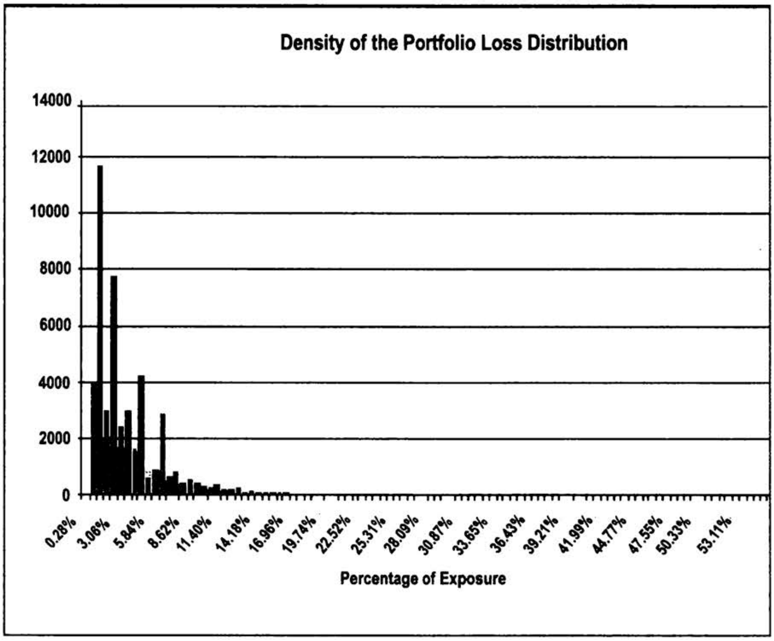

Figure 3: Histogram of the empirical loss distribution in a Monte-Carlo-Simulation

terparties, labeled by 1A to 40A. The loss distribution is given in Figure 3, below.

The Monte-Carlo-Simulation was based on a default-only approach. Since the portfolio is quite small, the discrete structure is still very noticeable. It is also obvious that the empirical loss distribution is even more skewed than the analytic approximation, cf. Figure 3 and Figure 2.

As capital definition the industry standard, i.e. the 99%-quantile of the loss distribution, is used. Within the Monte-Carlo-Simulation it was straightforward to evaluate risk contributions based on expected shortfall. The resulting risk contributions and its comparison to the analytically calculated risk contributions based on the volatility decomposition can be seen in Figure 4.

In the present portfolio example the difference between the contributory capital of two different types, namely analytic risk contributions and contribution to shortfall, should be noticed, since even the order of

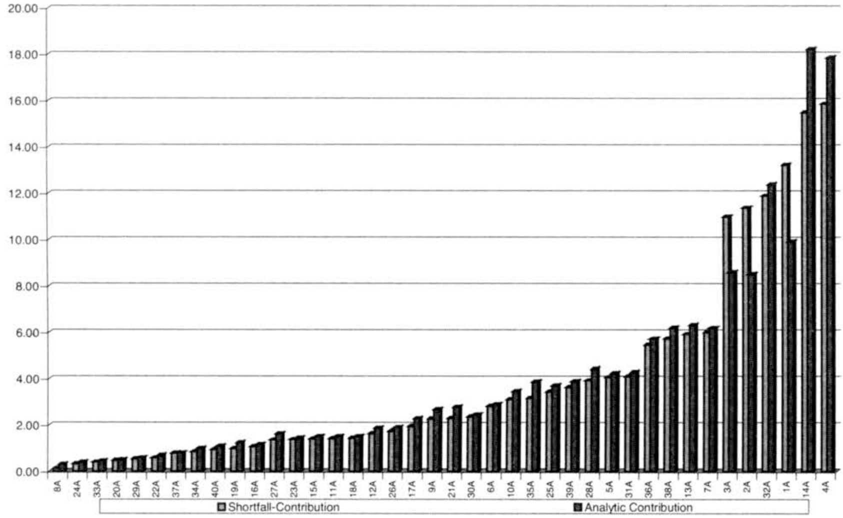

Figure 4: The bar charts depict the different risk contributions for every counterparty in the portfolio. The white columns belong to the counterparties' contribution measured by the shortfall, the black ones correspond to the analytic contribution.

the assets according to their risk contributions changed. The asset with the largest shortfall contributions, 4A, is the one with the second largest risk contribution and the largest risk contributions 14A goes with the second largest shortfall contribution. A view at the portfolio shows that the shortfall contributions are more driven by the relative asset size. However, it is always important to bear in mind that these results are still tied to the given portfolio.

It should also be noticed that the gradient of the EC is steeper for the analytic approach. Bad loans might be able to satisfy the hurdle rate in a RAROC-Pricing tool if one uses the expected shortfall approach, but might fail to earn above the hurdle rate if EC is based on Var/Covar.

Portfolio B

The second portfolio consists of 100 facilities mainly loans to large corporate customers. The expected loss of the portfolio is 18bp and the portfolio standard deviation 37bp. The capital allocation was based on the 99.9% quantiles which amounts to 266bp. The threshold in this case was set, such that the expected shortfall capital coincides with the 99.9%-EC.

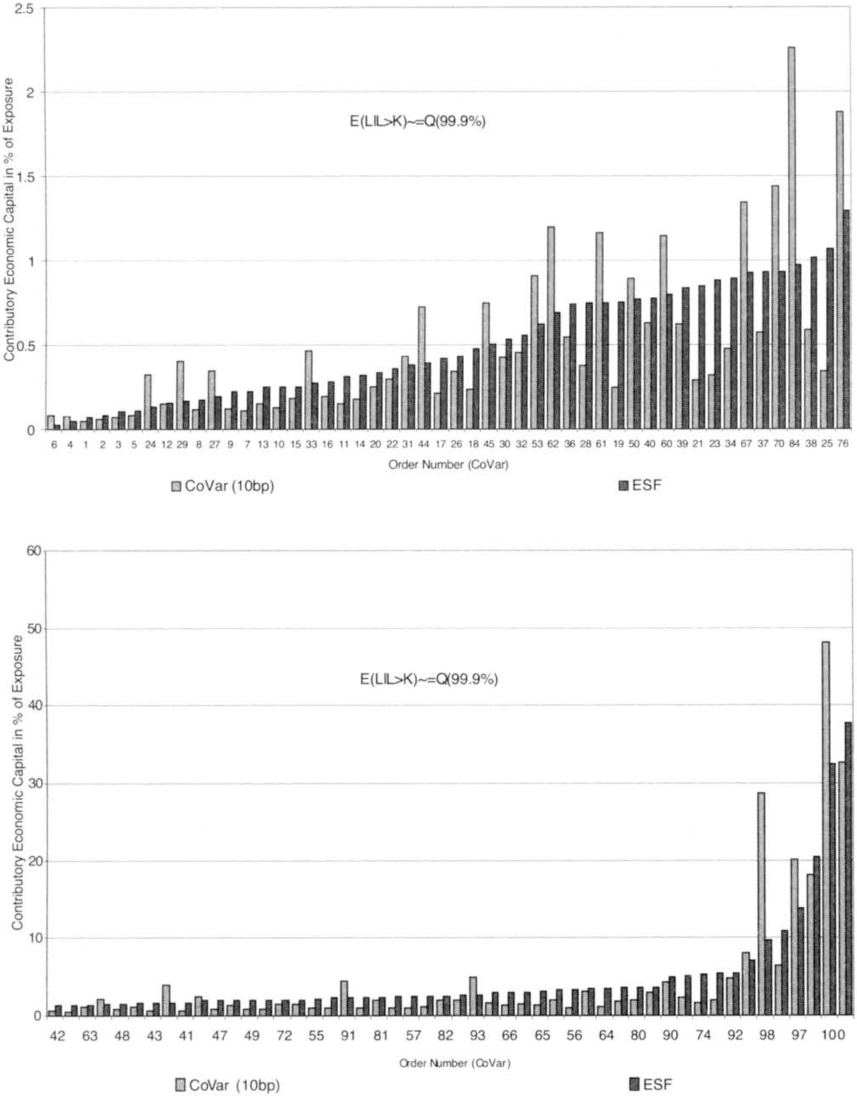

Figure 5: The bar charts depict the different risk contributions for every counterparty in the portfolio. The black columns belong to the counterparties' contribution measured by the shortfall, the white ones correspond to the analytic contribution.

The results displayed in Figure 5 are basically the same as for the smaller portfolio. In general, counterparties with high default probabilities get

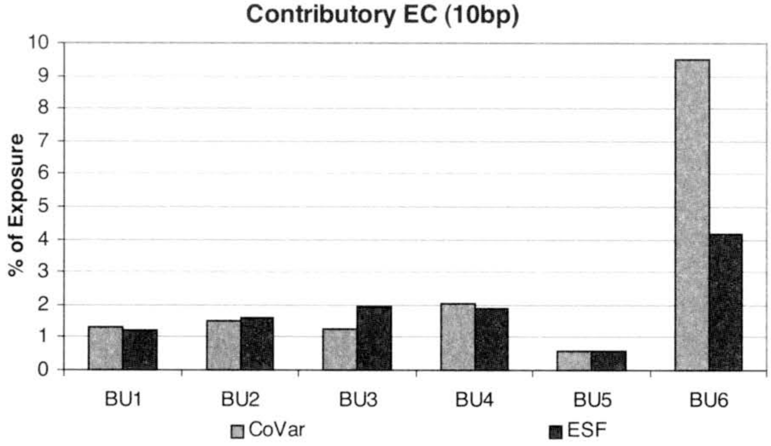

Figure 6: The bar charts depict the different risk contributions for every counterparty in the portfolio. The black columns belong to the counterparties' contribution measured by the shortfall, the white ones correspond to the analytic contribution.

a higher charge with the covariance allocation rule. The winners in this allocation rule are mostly those which small default probabilities, high exposures, or high correlation with the rest of the portfolio. Name concentration is better captured by shortfall contribution. If the capital allocation is driven by the desire to distribute cost for the insurance against catastrophic systemic risk the expected shortfall method exhibits its superiority. The covariance is mainly concentrated at the regular volatility.

Business Areas

Currently, the calculation of expected shortfall contribution requires a lot of computational power, which makes it less feasible for large portfolios. However, the capital allocation on business level can accurately be measured with expected shortfall contribution. Below in Figure 6 the reader finds an example of a bank with 6 business areas.

It is again obvious that there is a different allocation under expected shortfall. It is also known that some off-the-shelfes models have implemented risk capital allocation based on covariance. The banks might obtain results that their retail business requires less capital if they are not consolidated with the rest of the bank. This is mainly due to the fact that retail has high default probabilities and less name concentration. Therefore their loans are penalised by the covariance approach. Carrying out the capital allocation with expected shortfall lead to the result that it often doesn't matter if the retail business is part of a larger group or is capitalized independently, of course only from a credit risk capital point of view.

Therefore it is a good recommendation to think about a review of the standard methods to allocate contributory risk capital including the basic capital definition by Value-at-Risk.

Conclusions on Capital Allocation

At the present stage of research we recommend to use Expected Shortfall Contribution on aggregated levels in the portfolio structure. At the level with finest granularity, the single transactions, an allocation either based on variance/covariance or even on market spreads might be reasonable. On single facility level the pricing is important and volatility based measures seem suitable. The risk management of higher aggregated portfolios is more driven by a long term insurance type approach. Mainly the extreme downside risk should be measured and capitalized. Therefore the contribution based on expected shortfall fulfills this task better.

Additionally the computational difficulties of shortfall contribution on single transaction level can prevent the usage of this procedure on the finest aggregation level for large portfolios.

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Summary

Stochastic Essentials for the Risk Management of Credit Portfolios

Recent developments in portfolio and risk management are driven by the need of quantitative risk assessment. Mertons asset value approach is presented in a portfolio context. Loss distributions are derived and different definitions of economic capital are considered. In particular the loss distribution underlying the current Basel II discussions is derived. A challenging task for risk management is the allocation of the risk capital to business units and single transactions. An analysis of two capital allocation methods is carried out. One based on expected shortfall contribution in credit portfolio modeling and the other based on contribution to the volatility which is the more traditional one. It turns out that the second one overestimates the risk of low rated counterparties with low concentration risk. The reason for this is that at the standard deviation many small losses are important, whereas at the quantile of the loss distribution large but rare losses are more important. This is captured by Expected Shortfall. Therefore Expected Shortfall contribution rewards diversification – name, industry and regional diversification. (JEL G31, G24, G00)

Zusammenfassung

Stochastische Grundlagen für das Risikomanagement von Kreditportfolien

Neuere Entwicklungen im Portfolio- und Risikomanagement sind von der Suche nach quantitativen Modellen für die Risikobeurteilung motiviert. Mertons Firmenwertmodell wird in einem Portfoliokontext dargestellt. In diesem Zusammenhang werden Verlustverteilungen abgeleitet, insbesondere die Verlustverteilung, die in die gegenwärtige Basel-II-Diskussion einfließt. Eine wichtige Aufgabe des Risikomanagements ist die Zuordnung und Allokation des Risikokapitals auf Geschäftseinheiten und Einzeltransaktionen. Eine Analyse von zwei verschiedenen Allokationsmethoden wird durchgeführt. Die Erste basiert auf dem Konzept des erwarteten Verlustes bei großen Gesamtverlusten, Expected Shortfall Contribution, die Zweite beruht auf Beiträgen der Einzeltransaktion zur Volatilität der Verluste. Es

zeigt sich, dass Letztere das Risiko von Kreditnehmern mit kleinem Exposure und schlechtem Rating mit zu viel Kapital unterlegt, da die Volatilität stark von vielen Ausfällen kleiner Exposures beeinflusst wird. Für das Quantil sind die Ausfälle großer Exposures entscheidender. Dies wird vom Expected Shortfall erfasst. Deswegen reagiert eine Allokation basierend auf Expected Shortfall Contribution sensitiver auf Diversifikation und Konzentration, sowohl auf einzelne Namen als auch in Branchen und Regionen.

Résumé

Fondements stochastiques pour la gestion des risques et des portefeuilles de crédit

Les nouveaux développements de la gestion des portefeuilles et des risques cherchent à trouver des modèles quantitatifs pour évaluer les risques. Le modèle de la valeur des actifs de Merton est présenté dans un contexte de portefeuille. La distribution des pertes, en particulier, la distribution des pertes sous-jacente dans le modèle de Basel II, est dérivée. Une tâche importante de la gestion des risques est celle de l'allocation du capital à risque aux unités économiques et aux transactions simples. Une analyse de deux méthodes différentes d'allocation est faite ici. La première se base sur l'Expected Shortfall Contribution, le concept du risque attendu en cas de pertes globales importantes. La seconde se base sur les contributions des transactions simples à la volatitité des pertes. On voit que la deuxième méthode surestime le risque des emprunteurs avec une faible exposure et un mauvais rating, car la volatilité est fortement influencée par les pertes de petites exposures. Au quantile, les pertes de plus grandes exposures sont décisives. Ceci est enregistré par l'Expected Shortfall. C'est pourquoi, une allocation basée sur l'Expected Shortfall Contribution est plus sensible à la diversification et à la concentration – de noms, branches et régions.