On Black, Blue and Orange Bonds

A Tax Arbitrage Model with Asymmetric Taxation

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I. Introduction

This paper analyses a bond market that consists of differently taxed bonds (black, blue and orange). The taxation of the individual bond depends on the specific investor, whether he is a buyer or an issuer and upon the type of investor (private or corporate). Our analysis is based on the absence of after tax arbitrage opportunities for all investors. The literature on bond arbitrage has two main directions. The elimination of arbitrage opportunities is either done by a progressive tax system or by means of transaction costs and/or a (partial) ban on issuing bonds.

Dammon and Green's (1987) one-period, state model is an exponent of the first type of model. Their analysis considers necessary and sufficient conditions for arbitrage elimination solely by means of a progressive tax system. Dammon and Green (1987) looks at taxation of coupons and capital gains (one at a time). It follows directly from the paper of Dammon and Green (1987) that a progressive tax system, where zero is a possible marginal tax rate for each investor (i.e. tax reimbursement is not given in case of infinite negative taxable income), implies that the elimination of arbitrage opportunities can be done solely by the tax system. The consequences of such a type of model are absence of clienteles and a levelling of the marginal tax rates for coupon taxed investors (that is, investors taxable on coupon payments but tax-exempt on capital gains). If the future payments from the bonds span both the taxable as well as the nontaxable payments, a double spanned bond market, we obtain the result that all coupon taxed investors end up with the same

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marginal tax rate (Schaefer (1982), Dybvig and Ross (1986) and Dammon and Green (1987)). If we furthermore have investors that are also taxable on capital gains, the coupon taxed investors will have a common marginal tax rate of zero (Raaballe and Toft (1990)).[1]

The papers of Dermody and Prisman (1988), and Dermody and Rockafellar (1991) are examples of models of the latter type. In the last mentioned paper the elimination of arbitrage opportunities is done solely by means of transaction costs, as the investors' marginal tax rates are independent of taxable income. In the first paper we have a convex tax function, which is otherwise unspecified. The role of the tax system in the elimination of arbitrage opportunities is not analyzed. The consequence of this type of model is the existence of clienteles of the form, "investor will neither buy nor issue a particular bond" or "investor will not buy (issue) a particular bond".[2]

The present paper formulates a model that results in both identical marginal tax rates for private investors and creation of clienteles. Identical marginal tax rates are implied by the progressive tax system and a double spanned bond market. Private investors hereby obtain a common marginal tax rate that is no higher than the lowest marginal tax rate of the firms. In case of symmetric capital gain taxation of the corporate investors the private investors would be able to make arbitrage in such a way, that the common marginal tax rate for the private investors would end up being zero. This is, however, not possible here due to the asymmetric capital gain taxation. The asymmetric taxation in the present model has the same role in making clienteles as transaction costs and/or a ban on issuing bonds have in models such as those formulated by Dermody and Prisman (1988) and Dermody and Rockafellar (1991).

The model is, for the sake of simplicity, formulated within a one-period framework. However, the model can immediately be generalized into a multi-period framework. In a multi-period framework the tax authority becomes still more vulnerable to tax arbitrage activities. Section 5 of the paper provides an example of this.[3]

1 Ross (1987) can also be included in this group.

2 The papers of Schaefer (1981), Katz and Prisman (1991) and parts of Schaefer (1982) and Ross and Dybvig (1986) can also be included in this group.

3 The paper is also an analysis of a newly introduced tax system for bonds in Denmark. In the preliminary work on this tax system great importance has been put on removing/reducing the possibilities of making tax based profits. As the capital markets have become more competitive (especially lower transaction costs) investors have been taking long and short positions in fixed income bonds with the purpose of utilizing tax arbitrage possibilities. The reaction of the government

The paper is organized as follows. In section 2 we define the concepts and state the assumptions of the model. The model is set up in section 3, whereas its consequences are derived in section 4. The model is enlarged with previously issued blue bonds in section 5. Some concluding comments are offered in section 6. All proofs of theorems are relegated to the appendices.

II. Concepts and Assumptions

The taxable income of an investor is composed of the tax period's net coupon payments, taxable capital gains, tax-deductible capital losses and finally all other kinds of taxable income. Since we only consider risk-free, fixed income bonds with one year to maturity, the taxable income for all arbitrage portfolios will be known apart from the last term, that is, all other kinds of taxable income may not be known. If this last term is stochastic, e.g. wage, one should not form the arbitrage portfolio until later, i.e. the autumn where the taxable income for the period is almost certain. This will for example be the case if the wage per unit of time follows a diffusion process, which means that the wage income of the investor is locally deterministic. We will assume that all other taxable income is deterministic either now or later in the tax year. If all other taxable income only becomes deterministic later in the tax year, the arbitrage portfolio is made at that time. If this is the case the investors have to make arbitrage portfolios at a much larger scale, making the model more vulnerable to transaction costs.

The tax payment of a particular investor falls due at the end of the period and is a non-decreasing convex function of the taxable income. All marginal tax rates of the investors belong to the interval $[0,1[$. For every investor there exists a taxable income (perhaps negative) where his marginal tax rate is zero. If the tax function is not differentiable for all arguments, an element from the set of subgradients is used as the marginal tax rate. The tax functions of the investors are allowed to be mutually different.

The coupon payment and the face-value of a bond at time 1 are denoted as c and f. Accordingly the coupon rate of a bond is denoted by

to this was first to level the marginal tax rates for different groups of investors, and now to introduce a tax system with asymmetric taxation, as a relatively low taxation of firms is considered desirable. As this paper shows, the arbitrage opportunities do not vanish completely. Elimination of the arbitrage opportunities (disregarding the transaction costs) gives rise to common marginal tax rates for private investors at the same level as the corporate tax rate (if foreign investors are taken into account, we get an even more drastic picture).

c/f. All bonds which at the time of issue have a coupon rate $c/f \in [c_*/f_*, c_{**}/f_{**}]$ are defined as blue bonds. c_{**}/f_{**} is chosen in such a way that a newly issued blue bond with this coupon rate sells at a price equal to face-value. c_*/f_* is chosen by the tax authority such that it is lower than c_{**}/f_{**} (according to Danish tax laws c_*/f_* is typically fixed such that it is two percentage points below c_{**}/f_{**}). All bonds which at the time of issue have a coupon rate above c_{**}/f_{**} are defined as orange bonds. All bonds which at the time of issue have a coupon rate of $c/f \in [0, c_*/f_*[$ are defined as black bonds. The relations are illustrated in figure 1.

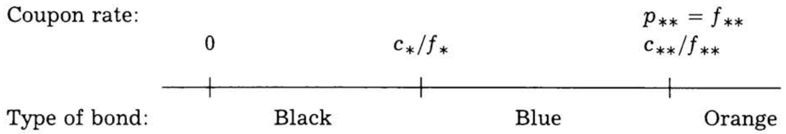

Figure 1

Coupon payments are taxed symmetrically for all types of investors. The taxation of capital gains/losses depends on the type of investor. The rules are as follows:

- For corporate investors capital gains/losses are, as a principal rule, taxed symmetrically. The exception concerns short positions in blue and orange bonds where capital losses are not tax-deductible.
- For private investors capital gains/losses are, as a principal rule, tax-exempt. The exception concerns capital gains on black bonds and capital gains from short positions in orange bonds, both types of capital gains are taxable.

By tax arbitrage for an investor we mean a bond portfolio which generates non-negative after tax payments at time 0 as well as at time 1. Tax arbitrage can be divided into finite and infinite tax arbitrage. An infinite tax arbitrage is a tax arbitrage which generates infinite after tax payments at time 0 or at time 1. In the sections to follow we derive the consequences of absence of infinite tax arbitrage opportunities for all investors. In this model an infinite tax arbitrage at time 1 can be transformed into an infinite tax arbitrage at time 0. This is done by a bond issue at time 0. It is therefore sufficient to preclude infinite tax arbitrage at time 0.

III. The Model

Without loss of generality we can restrict the analysis to 4 bonds: The blue bond with the largest coupon rate (no. 1), the blue bond with the smallest coupon rate (no. 2), a black bond (no. 3) and an orange bond (no. 4). The notation of the model is as follows

f_i $\equiv$ Principal payment (face-value) of bond no. i at time 1

c_i $\equiv$ Coupon payment of bond no. i at time 1

p_i $\equiv$ Price of bond no. i at time 0

x_i^l $\equiv$ Number of bonds no. i bought at time 0

x_i^s $\equiv$ Number of bonds no. i issued/sold at time 0

O_p/O_v $\equiv$ All other taxable income of a private/corporate investor at time 1

$T_p(\cdot)/T_v(\cdot)$ $\equiv$ Tax payment for a private/corporate investor at time 1 as a function of the period's taxable income

t_p/t_v $\equiv$ The marginal tax rate of a private/corporate investor at a given taxable income.

By definition of the bonds we have

$$\text{(A1)} \qquad c_3/f_3 < c_2/f_2 = c_*/f_* < c_{**}/f_{**} = c_1/f_1 < c_4/f_4 \quad p_1 = f_1$$

Later it will be shown that (A1) is equivalent to

$$\text{(A2)} \qquad p_3/f_3 < p_2/f_2 < p_1/f_1 < p_4/f_4 \quad p_1 = f_1 \text{ and } c_3/f_3 < c_2/f_2$$

Absence of infinite tax arbitrage opportunities is most easily characterized by means of (A2).

The tax arbitrage problem of a private investor can be stated as follows

$$\text{(PP)} \begin{cases} \text{Min } p_1 x_1 + p_2 x_2 + p_3 (x_3^1 - x_3^s) + p_4 (x_4^1 - x_4^s) \\ \text{s.t.} \\ x_1 (f_1 + c_1) + x_2 (f_2 + c_2) + (x_3^1 - x_3^s)(f_3 + c_3) + (x_4^1 - x_4^s)(f_4 + c_4) \\ -T_p (O_p + x_1 c_1 + x_2 c_2 + x_3^1 (c_3 + f_3 - p_3) \\ - x_3^s c_3 + x_4^1 c_4 - x_4^s (c_4 + f_4 - p_4)) \geq -T_p (O_p) \quad (d) \\ x_3^1, x_3^s, x_4^1, x_4^s, \geq 0 \end{cases}$$

(PP) seeks to form a portfolio which generates a nonnegative additional after tax payments at time 1 at the minimal cost. A tax arbitrage portfolio exists as $\underline{x} = \underline{0}$ is a feasible solution to (PP).

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The restrictions are explained as follows. A private investor is taxed symmetrically of all blue bonds (no. 1 and 2) and only of the coupons. For a long position in black bonds (no. 3) coupon payments as well as capital gains ($f_3 > p_3$) are taxable, whereas only coupons are tax deductible for a short position in the same bonds. Coupons from positions in orange bonds (4) are taxed symmetrically, and in addition, capital gains from short positions in orange bonds accruing to the private investor are taxable. This is the case in this analysis since $p_4 > f_4$.

From the theory of convex analysis[4] we have, that absence of infinite tax arbitrage for the private investor is equivalent to the existence of a $d > 0$ satisfying the conditions (3) - (6a) stated in appendix 1.

The tax arbitrage problem of a corporate investor can be stated as follows

$$
(\mathrm{PV})\left[\begin{array}{l}
\mathrm{Min}\ p_1 x_1 + p_2 (x_2^l - x_2^s) + p_3 x_3 + p_4 x_4 \\
\mathrm{s.t.} \\
x_1 (f_1 + c_1) + (x_2^l - x_2^s)(f_2 + c_2) + x_3 (f_3 + c_3) + x_4 (f_4 + c_4) \\
-T_v (O_v + x_1 (c_1 + f_1 - p_1) + x_2^l (c_2 + f_2 - p_2) - x_2^s c_2 + x_3 (c_3 + f_3 - p_3) \\
+ x_4 (c_4 + f_4 - p_4)) \geq -T_v (O_v) \quad (d_v) \\
x_2^l,\ x_2^s,\ \geq 0
\end{array}\right.
$$

Absence of infinite tax arbitrage for the corporate investor is equivalent to the existence of a $d_v > 0$ satisfying the conditions (1) - (2a) stated in appendix 1.

The restrictions are explained as follows. Since $p_1 = f_1$ the corporate investor will, in fact, be taxed symmetrically on the entire return from the blue bond with the largest coupon rate. For long positions in bond no. 2 the entire return is taxable for the corporate investor. On short positions capital losses are not tax-deductible. This explains the tax treatment of bond no. 2 (since $f_2 > p_2$). Black bonds are taxed symmetrically on the basis of the entire return. Due to the fact that $p_4 > f_4$, a short position in the orange bond gives rise to a taxable capital gain, and this bond is therefore taxed symmetrically for the firm.

Absence of infinite tax arbitrage opportunities for corporate and private investors is equivalent to the existence of a $d > 0$ and a $d_v > 0$ satisfying the conditions (1) - (6a) stated in appendix 1.

[4] For a more detailed account of this topic within the field of arbitrage pricing see Ross (1987) and/or Dammon and Green (1987).

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In stating the problems (PP) and (PV) we needed assumption (A2). In appendix 2 it is justified that assumption (A2) is equivalent to assumption (A1).

After these preliminary exercises we are now able to state and discuss the results.

IV. Results and Analysis in the Case without Old, Blue Bonds

After the presentation of a result, the economic support for the result is given.

Proofs of all theorems are placed in appendix 3. The survey of results is followed by a discussion of the role of blue bonds in the arbitrage analysis. The section is concluded with an example.

Theorem 1: Absence of infinite tax arbitrage opportunities for the private investors implies that all private investors have the same marginal tax rate – i.e. $t_p = t_*$ for all p.

For private investors the blue bonds with large (no. 1) and small (no. 2) coupon rates are taxed symmetrically. Bond no. 1 yields the entire return as a taxable interest payment, whereas bond no. 2 yields part of its return as a tax exempt capital gain. If these two bonds have the same before tax return, all the private investors having positive marginal tax rates will prefer to have a long position in bond no. 2 and a short position in bond no. 1. If the private investor, who is assumed to have a positive tax rate, t_*, is to be indifferent between the two bonds, then bond no. 2 should have a lower before tax return. For investors with high tax rates ($t_p > t_*$) bond no. 2 then yields a higher after tax return than bond no. 1. Consequently, these investors will buy bond no. 2 and issue bond no. 1. Similarly, investors having a low tax rate ($t_p < t_*$) will engage in the opposite transaction. In this way high and low taxed investors engage in a mutual advantageously finite tax arbitrage; the high taxed private investor takes a long position in blue bonds having a low coupon rate and a short position in blue bonds having a high coupon rate. The low taxed investors take the opposite positions. The tax arbitrage implies that the taxable income of the high (low) taxed investor is decreased (increased). By this, the marginal tax rate of the high (low) taxed investor is decreased (increased). The arbitrage opportunities are not exhausted until all private investors have the same marginal tax rate (t_*).

Theorem 2: Absence of infinite tax arbitrage opportunities for all investors implies

a) The after tax return of the firm depends on v and is equal to $r(1 - t_v)$, where r is the before tax return defined by $1 + r = (f_1 + c_1)/p_1 = (f_3 + c_3)/p_3 = (f_4 + c_4)/p_4$ $(\geq (f_2 + c_2)/p_2)$.

b) The after tax return of the private investors is equal to $r(1 - t_*)$, where t_* is the common marginal tax rate of the private investors.

Furthermore, if the common marginal tax rate of the private investors (t_*) is strictly positive we have

c) $r > r_2$, where r_2 is defined by $1 + r_2 = (f_2 + c_2)/p_2$

d) Black bonds can only enter as a long position in the portfolios of the private investors

e) Orange bonds can only enter as a short position in the portfolios of the private investors.

Theorem 3: Absence of infinite tax arbitrage opportunities for all investors implies

a) For the common marginal tax rate of the private investors we have $0 \leq t_* \leq \min_v t_v$

Furthermore, if the common marginal tax rate of the private investors (t_*) is strictly positive we have

b) Bond no. 2 can only enter the portfolios of the firms as a short position – and in that case only for firms characterized by $\min_v t_v = t_*$.

In appendix 3 it is shown, that the taxable income of a private investor decreases when he engages in an arbitrage portfolio where bond no. 2 enters long and bond no. 1 or no. 4 enters short. On the other hand the taxable income of a firm increases when the firm engages in the opposite arbitrage portfolio.

As a proposition to theorem 3 it is noticed that all private investors will have a common marginal tax rate of zero if there exists a firm such that $t_v = 0$. This is the case either if some firms are totally free of taxation or if some firms does not pay tax on income up to a certain limit. If this limit is reached, by transferring taxable income from private investors to firms, new firms will be established for arbitrage purposes.

Let $\hat{t}_p$ be defined as the common marginal tax rate for the private investors if their aggregated taxable income, before tax arbitrage, was distributed among private investors in a way that would make their marginal tax rates equal.

Theorem 4: Absence of infinite tax arbitrage opportunities for all investors implies

a) If $\hat{t}_p > \min_v t_v$ then $t_p = t_* = \min_v t_v$ for all private investors

b) If $\hat{t}_p < \min_v t_v$ then $t_p = \hat{t}_p = t_* < \min_v t_v$ for all private investors

All private investors have the same marginal tax rate. If this rate is larger than the corporate marginal tax rate it can be reduced to the latter level by performing arbitrage. All economic units thus have identical marginal tax rate – i.e. we have de facto a common linear tax system. If the common marginal tax rate of the private investors is less than the lowest marginal corporate tax rate, then only the private investors will have a common linear tax system. In this case, however, it is difficult to see what role there is left for the firms to fulfil.

The economic reasoning behind the above results is as follows. There are two cases.

A. The aggregated taxable income of all persons, distributed in a way such that all private investors have a common marginal tax rate, implies a marginal tax rate for private investors which is larger than the smallest marginal corporate tax rate.

We have previously argued that all private investors obtain the same marginal tax rate. We can therefore treat all the private investors as one single investor. A corporate investor is taxable on the entire return from a long position in bond no. 1. By an issue of bond no. 2 the corporate investor has to pay a return, which is only partly tax deductible. Accordingly, the before tax return on bond no. 2 will have to be lower than the before tax return on bond no. 1 for the corporate investor to enter an arbitrage portfolio composed of a long position in bond no. 1 and a short position in bond no. 2. Since the common marginal tax rate of the private investors is larger than the lowest marginal corporate tax rate, the private investors will benefit from entering the opposite portfolio. By trading such portfolios the aggregated taxable income of the corporate investors is increased to the same degree as the aggregated taxable income of the private investors is decreased. The arbitrage stops when the common marginal tax rate of the private investors is equal to the lowest marginal corporate tax rate.[5]

[5] The arbitrage portfolio increases the taxable income of the firm. If the firms tax function is not linear for all positive taxable incomes, the arbitrage portfolio

For firms having the lowest marginal tax rate it is a matter of indifference whether an issue of bonds is done by means of bond no. 1 or 2. Since all of the before tax return from bond no. 1, contrary to bond no. 2, is tax deductible, corporate investors with a higher marginal tax rate will never be short in bond no. 2.

As the before tax return on bond no. 2 is lower than that of bond no. 1 (at a positive marginal tax rate for the private investors) and firms are taxed on capital gains from long positions, the corporate investors will never hold a long position in bond no. 2.

Due to the taxation of corporate investors, the before tax returns of bond no. 1, 3 and 4 are identical and accordingly a private investor, assuming a positive marginal tax rate, will never hold a short position in black bonds (no. 3) nor a long position in orange bonds (no. 4). This follows from the fact that on a before tax basis it costs the same to issue bond no. 1 and no. 3, and in addition, bond no. 1 gives the largest tax reduction. Since bond no. 1 and no. 4 yields the same before tax return, and the latter gives the largest taxable income, it is unfavorable for any private investor to have a long position in bond no. 4.

B. *The aggregated taxable income of all persons, distributed in a way such that all private investors have a common marginal tax rate, implies a marginal tax rate for private investors which is less than the smallest marginal corporate tax rate.*

Again we treat the private investors as a single investor. Is it possible for the private and corporate investors to engage in tax arbitrage in such a way as to equalize the marginal tax rate of private investors and the lowest corporate marginal tax rate? In that case the corporate investor, who is the high taxed investor, should hold a long position in bond no. 2 and a short position in bond no. 1. Since the entire return accruing to the corporate investors from bond no. 2 is taxable, this implies that the before tax return of the two bonds should be equal, but then it would not be profitable for the private investors to enter in the opposite positions due to a before tax liquidity of zero and a taxable surplus. The opposite position does not work either, since breakeven for the corporate investor will result in a negative liquidity for the private investor. The conclusion of this case is that absence of infinite tax arbitrage implies a

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will cause the corporate investors marginal tax rate to rise. Instead of accepting this increased corporate tax rate we could introduce, if possible, new firms of the same type.

common personal marginal tax rate, which is lower than the smallest corporate marginal tax rate.

We have seen that it never will be profitable for the firm to have a short position in bond no. 2 combined with a long position in bond no. 1. Consequently, the firm will not borrow by means of bond no. 2. Similarly the firm will not hold a long position in bond no. 2 either, as the taxation of the private investors (if positive tax rates) implies that bond no. 2 has a lower before tax return than all other bonds. Since long positions are taxed equally for firms, independent of the color of the bond, an investment in bond no. 1 dominates an investment in bond no. 2. Accordingly, a firm will neither hold a long nor a short position in bond no. 2.

By analogy with the previous cases the private investors will never, if they have a positive marginal tax rate, hold a short position in black bonds nor a long position in orange bonds.

The tax arbitrage between private and corporate investors in case A is exploited by means of blue bonds. It does not matter, however, if the blue bond with the largest coupon rate is replaced by an orange bond. This is due to the fact that the entire return from a short position in this bond is tax deductible for the private investor (capital gains from short positions in orange bonds are taxable for private investors), while the entire return accruing from a long position in the orange bonds are taxable for the firms. The orange bond therefore has the same role in the tax arbitrage arguments as the blue bond with the largest coupon rate.

From the above analysis it follows that the conclusions would have been exactly the same, had the blue bond with the largest coupon rate been defined by a coupon rate, which implies a price below face-value. If, on the other hand, the blue bond having the largest coupon rate have had a price above face-value, then our analysis would have been completely different. In that case firms could form a tax arbitrage portfolio by taking a long position in the blue bond with the largest coupon rate and a short position in the black bond. Since the entire return accruing from the long as well as from the short position is taxable for the corporate investor, the absence of infinite arbitrage opportunities implies, that the before tax return is equal for the two bonds. By entering the opposite portfolio private investors achieve a before tax gain of zero and a tax deductible deficit. This portfolio generates, in other words, a positive liquidity until the marginal tax rate of the private investors is reduced to zero. This means that private investors may have long as well as short positions in all kinds of bonds, since all the bonds will have the same before tax return. It is thus only blue bonds having low coupon rates ($p < f$) that exhibit clienteles. An equivalent tax arbitrage could have been created using two blue bonds, both with prices above face-value.

Example: In Denmark the taxation of the private and corporate investors are as shown in figure 2.

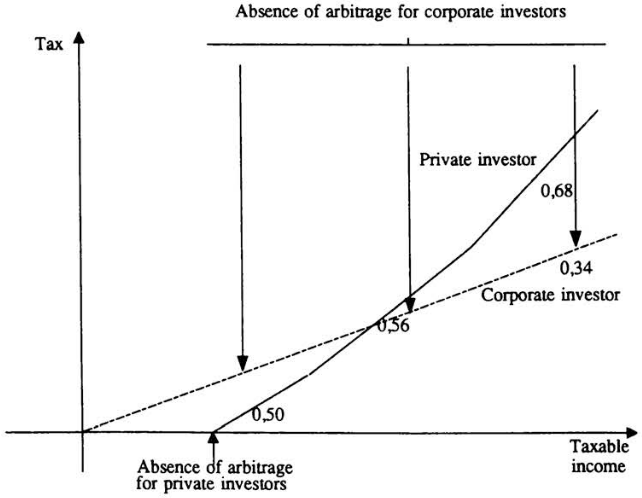

Figure 2

Since the aggregated income of the private investors conditions a $\hat{t}_p > 0{,}34$, the absence of infinite arbitrage opportunities implies the following drastic conclusions

- Private investors do not pay tax (directly).
- $t_p = t_* = 0{,}34$. Private investors are taxed indirectly at the corporate tax rate.
- The private investors (considered as a group) hold an arbitrage portfolio, which is long in bond no. 2 and short in bond no. 1 (or no. 4). The corporate investors hold the opposite portfolio.
- The private investors will neither hold orange bonds nor issue black bonds.
- The corporate investors will never hold blue bonds having low coupon rates.

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V. Results and Analysis in the Case with Old, Blue Bonds

Motivated by the discussion in the previous section about the arbitrage potential of blue bonds having high coupon rates we now enlarge the model by adding previously issued blue bonds (no. 0). We will assume that $c_0/f_0 > c_1/f_1$. It should be emphasized that bond no. 0 is unique in that way that we cannot issue a new bond with a coupon rate larger than c_1/f_1 and at the same time have it blue printed. For that reason an investor cannot sell (issue) more than his existing stock $(\bar{x}_0^1)$ of old blue bonds (no. 0) at time zero.

The tax arbitrage problem of a private investor is modified as follows

$$
(\text{MPP})\left[\begin{array}{l}
\text{Min } p_1 x_1 + p_2 x_2 + p_3 (x_3^1 - x_3^s) + p_4 (x_4^1 - x_4^s) + p_0 (x_0^1 - x_0^s) \\
\text{s.t.} \\
x_1 (f_1 + c_1) + x_2 (f_2 + c_2) + (x_3^1 - x_3^s)(f_3 + c_3) + (x_4^1 - x_4^s)(f_4 + c_4) \\
+ (x_0^1 - x_0^s)(f_0 + c_0) - T_p (O_p + x_1 c_1 + x_2 c_2 + x_3^1 (c_3 + f_3 - p_3) - x_3^s c_3 \\
+ x_4^1 c_4 - x_4^s (c_4 + f_4 - p_4) + (x_0^1 - x_0^s) c_0) \geq - T_p (O_p) \quad (d) \\
\bar{x}_0^1 (p) - x_0^s \geq 0 \;\; (\lambda) \\
x_3^1,\, x_3^s,\, x_4^1,\, x_4^s,\, x_0^1,\, x_0^s \geq 0
\end{array}\right.
$$

The modifications are explained as follows. A private investor is taxed symmetrically of all blue bonds (here no. 0) and only of the coupons. Since c_0/f_0 is larger than the maximum coupon rate for blue bonds an investor cannot issue (i.e. sell) more than $\bar{x}_0^1(p)$ of the previously issued bonds; a new issue of such bonds will be characterized as orange bonds. From the theory of convex analysis we have that absence of infinite tax arbitrage opportunities for the private investor is equivalent to the existence of a $d > 0$ satisfying the conditions (3) - (6b) stated in appendix 4.

The tax arbitrage problem of the corporate investor is modified in a similar way. A long position in bond no. 0, which is a blue printed bond, is taxed in the same way as bond no. 1 and 2. A corporate investor may also sell from his existing stock of bond no. 0. But, the analysis in this section shows that the private investors will never advantageously buy bond no. 0. Hence, we omit sales[6] of bond no. 0 from the program of the

[6] For the sake of completeness the sale of bond no. 0 of the corporate investor is discussed here. Two cases are to be considered: Capital gains taxation is based on i) an accrual basis or ii) on the basis of realization. In case i) the tax basis of the existing stock of bond no. 0 is p_0 at time 0, hence the liquidity of a sold position is symmetric to the liquidity of a bought position. In case ii) the liquidity from a

corporate investor. The modified tax arbitrage problem of a corporate investor is as follows

$$
(\text{MPV})-\left[\begin{array}{l}
\text{Min } p_1 x_1 + p_2 (x_2^1 - x_2^s) + p_3 x_3 + p_4 x_4 + p_0 x_0^1 \\
\text{s.t.} \\
x_1 (f_1 + c_1) + (x_2^1 - x_2^s)(f_2 + c_2) + x_3 (f_3 + c_3) + x_4 (f_4 + c_4) + x_0^1 (f_0 + c_0) \\
-T_v (O_v + x_1 (c_1 + f_1 - p_1) + x_2^1 (c_2 + f_2 - p_2) - x_2^s c_2 \\
+x_3 (c_3 + f_3 - p_3) + x_4 (c_4 + f_4 - p_4) + x_0^1 (c_0 + f_0 - p_0)) \geq -T_v (O_v) \ \ (d_v) \\
x_2^1, x_2^s, x_0^1, \geq 0
\end{array}\right.
$$

Absence of infinite tax arbitrage opportunities for the corporate investor is equivalent to the existence of a $d_v > 0$ satisfying the conditions (1) - (2b) stated in appendix 4. In appendix 4 we also prove that $c_0/f_0 > c_1/f_1$ is equivalent to $p_0/f_0 > p_1/f_1 = 1$.

The theorems of this section are proved in appendix 5. In the remaining part of this section we will focus on an intuitive presentation and reasoning.

As the coupon rate of bond no. 0 is larger than the one of bond no. 1, the bond no. 0 is unique in the sense that the supply cannot be increased since newly issued bonds having the same coupon rate would be labelled orange. The results in this section crucially depend on the aggregated stock of bond no. 0 that the private investors initially possess. Selling out of the stock is the only way in which private investors can issue (shortsell) bond no. 0.

From the previous section it is known that the elimination of tax arbitrage opportunities by means of bond no. 1 and 2 implies identical marginal tax rates for the private investors. We can thus treat the group of private investors as a single private investor.

If our private investor forms a portfolio composed of a long position in bond no. 1 and a short position in bond no. 0 we need a corporate inves-

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sold position will depend on the tax basis of the sold position at time 0, p_*. Define $\varepsilon_v = (p_0 - p_*)\,(t_v^0 (1 + r_1 (1 - t_v^1)) - t_v^1)$, where t_v^0/t_v^1 are the marginal tax rates of the corporate investor at time 0 and 1 and $r_1 = c_1/f_1$ is the before tax return. If $\varepsilon_v < 0$ it can be shown that it is optimal for the corporate investor to sell and buy back at time 0 some or all of his existing stock of bond no. 0 – i.e. a wash-sale is executed. If $\varepsilon_v \geq 0$ a wash-sale is not optimal. In both cases the group of private investors is not involved. Hence, the only advantage of modeling the sales of bond no. 0 of the corporate investors is to illuminate the wash-sale proces of the corporate investors in case that the capital gains taxation is based on a principle of realization.

tor to enter in the opposite portfolio. Since capital gains/losses from a long position in bond no. 0 as well as from a short position in bond no. 1 de facto are taxable for the corporate investors, the before tax return of the two bonds for the corporate investor must be equal. The opposite portfolio is thus an arbitrage portfolio that does not change the liquidity at the two points in time. The portfolio is an arbitrage portfolio for the private investor since it on a before tax basis generates a liquidity of zero and gives rise to a tax deductible deficit (since bond no. 0 has a higher coupon rate than its before tax return). The tax arbitrage for the private investor stops either when the person does not have any more bonds to sell or when the reduction in the taxable income implies a marginal tax rate of zero.

In characterizing the cases below we define

$TI \equiv$ The taxable income of the private investor before arbitrage with the firms minus the tax deficit from selling all of bond no. 0 and buying bond no. 1 for the proceeds.

There are three cases to consider:

A. Ample. TI conditions a personal marginal tax rate of zero

The tax arbitrage consists of the private investor selling bond no. 0 and taking a long position in bond no. 1 with a corporate investor as counterpart. The corporate investor is in breakeven since the before tax returns are identical and the firm is taxed on basis of the entire return of the bonds.

As the tax deductible deficit from bond no. 0 is larger than the taxable income from bond no. 1 the private investor makes a profit from the arbitrage as long as his marginal tax rate is above zero. The assumption A implies that the arbitrage can be accomplished in such a scale as to reduce all the marginal tax rates of the private investors to zero.

Due to the common marginal tax rate of zero for all private investors, these may have long as well as short positions in all kinds of bonds.

The before tax returns of bond no. 1, 2 and 0 are identical due to the pricing based on the marginal tax rate of zero for all private investors. Since the return on bond no. 2 consists of an interest payment as well as a capital gain, it is cheaper for the firm to borrow by means of bond no. 1. The firm would therefore never issue bond no. 2.

It is noticed that the above arbitrage reduces the aggregated taxable income of the private investors without increasing the taxable income of the corporate investors.

B. Scarce. TI conditions a personal marginal tax rate, which is larger than zero but less than the smallest corporate marginal tax rate

The tax arbitrage arrangement is similar to the one under item A, but it now stops when the person has sold his entire stock of bond no. 0. The private investors thus achieve, a common positive marginal tax rate, which is less than the smallest corporate marginal tax rate.

Since the private investors have a positive marginal tax rate it would have been profitable for them to continue the arbitrage, if possible, by means of bond no. 0 and 1. Accordingly they will not have a long position in bond no. 0. From item B of section 4 we have that no private investor will ever issue black bonds or hold long positions in orange bonds. Furthermore, we have that corporate investors never will have any positions in bond no. 2.

C. Very scarce. TI conditions a personal marginal tax rate, which is larger than or equal to the smallest corporate marginal tax rate

Again the tax arbitrage follows the scheme set forth under item A, but now it has to stop at a marginal personal tax rate, which is larger than the smallest marginal corporate tax rate. However, the tax arbitrage opportunities are not exhausted by this since the private and corporate investor (as previously) can perform arbitrage by means of bond no. 1 and 2 until the common marginal tax rate of the private investors is equal to the smallest marginal corporate tax rate. (The first mentioned arbitrage opportunity is of course the most profitable as it implies a reduction in the taxable income of the private investors without increasing the taxable incomes of the firms. The last mentioned arbitrage opportunity implies that the firm is compensated for the increased taxable income through a lower return on bond no. 2. The advantage of the private investors in decreasing their taxable income is thus diminished by a lower return on the long asset (2)).

The clientele effects are as under item A of section 4. Moreover, no private investor will ever have a long position in bond no. 0.

The analysis in this section of tax arbitrage opportunities and clienteles as a function of the initial stock of old, blue bonds is, of course, consistent with the considerations in section 4 of the consequences of allowing agents to issue blue bonds at a price above face-value. This conforms in reality to the case A in this section. If the arbitrageur is precluded from issuing new blue bonds at a price above their face-value, he may initially try to secure himself a large stock of old blue bonds with a current price above their face-value. At a particular point in time the private investor issues a large number of bonds at par – for example some bonds that expires ultimo the year – and buys these himself. Such bonds are then forever classified as blue bonds. This portfolio is (apart from transaction costs) cost free for the private investor since long and short positions in blue bonds are taxed symmetrically for the private investor. If the investor later ascertains that one of these bonds with maturity this year is selling at a price above its face-value he sells some of these bonds and buys instead new bonds at par with the same time to maturity. The investor has hereby created a tax arbitrage portfolio as described in case A of this section. If the investor at every point in time forms these cost free portfolios of bonds at par, he will – if the price movements are suitable stochastic (for example follows a diffusion process) – be almost certain to end up with this opportunity each and every tax year. It should be noticed that the above strategy is very vulnerable with respect to transaction costs.

VI. Concluding Remarks

In this paper a model with asymmetric taxed bonds is set up and analyzed. It is shown that the absence of infinite tax arbitrage opportunities implies that all private investors will end up with the same marginal tax rate. This tax rate is no larger than the smallest corporate marginal tax rate. At the same time all types of bonds except claims selling at par exhibits clienteles. In the literature on bond arbitrage the clienteles are created by means of transaction costs or a (partial) ban on issuing bonds. The exhibition of clienteles in this model hinges on asymmetric taxation. The paper is also an analysis of the tax code governing the danish bond market. By continuously issuing claims at par and keeping them until their prices are above their face values, the private investors are able to achieve a marginal tax rate of zero. However, the last conclusion is very vulnerable to transaction costs.

Appendix 1

Absence of infinite tax arbitrage opportunities for private and corporate investors is equivalent to the statement that d and $d_v > 0$ satisfy the following set of equalities and inequalities. d and $\hat{d}$ are denoted without the subscript p. The reason is stated in theorem 1 in section 4.

(1) $$p_i = (f_i + c_i)\,\hat{d}_v \quad i = 1,\ 3 \text{ and } 4$$

(2a) $$[f_2 + c_2(1 - t_v)]\,d_v \geq p_2 \geq (f_2 + c_2)\,\hat{d}_v$$

$$\hat{d}_v \equiv \frac{(1 - t_v)\,d_v}{1 - t_v\,d_v}$$

(3) $$p_1 = [f_1 + c_1(1 - t_p)]\,d \ \text{ or } \ p_1 = (f_1 + c_1)\,\hat{d} \quad (\text{since } f_1 = p_1)$$

(4) $$p_2 = [f_2 + c_2(1 - t_p)]\,d$$

(5) $$[f_3 + c_3(1 - t_p)]\,d \geq p_3 \geq (f_3 + c_3)\,\hat{d}$$

(6a) $$[f_4 + c_4]\,\hat{d} \geq p_4 \geq [f_4 + c_4(1 - t_p)]\,d$$

$$\hat{d} \equiv \frac{(1 - t_p)\,d}{1 - t_p\,d}$$

From (1) (for $i = 1$), (3) and $d > 0$ it follows that $\hat{d} = \hat{d}_v > 0$

Appendix 2

In this appendix it is proved that (A1) and (A2) is equivalent.

It is noticed that $c_3/f_3 < c_2/f_2$ and $p_1 = f_1$ are common to both assumptions.

(A2) ⇒ (A1)

i) $$p_2/f_2 < 1 = p_1/f_1 \Rightarrow c_2/f_2 < c_1/f_1$$

c_1/f_1 is defined such that $p_1/f_1 = 1$. Accordingly we have from (3) and (4)

$$\frac{p_2}{f_2} = \frac{1 + \frac{c_2}{f_2}(1 - t_p)}{1 + \frac{c_1}{f_1}(1 - t_p)}. \text{ From this i) follows.}$$

ii) $$p_1/f_1 = 1 < p_4/f_4 \Rightarrow c_1/f_1 < c_4/f_4$$

From (1) for $i = 1$ and 4 we have

$$\frac{p_4}{f_4} = \frac{1 + \frac{c_4}{f_4}}{1 + \frac{c_1}{f_1}}. \text{ From this ii) follows.}$$

(A1) ⇒ (A2)

i) $$c_3/f_3 < c_2/f_2 \Rightarrow p_3/f_3 < p_2/f_2$$

In establishing (1) (for $i = 3$) and the right hand side of (2a) we did not use assumption (A2). Hence, from these two expressions we have

$$\frac{p_2/f_2}{p_3/f_3} \geq \frac{1 + \frac{c_2}{f_2}}{1 + \frac{c_3}{f_3}} > 1, \text{ since } c_3/f_3 < c_2/f_2. \text{ From this i) follows}$$

ii) $$c_2/f_2 < c_1/f_1 \Rightarrow p_2/f_2 < 1 = p_1/f_1$$

In establishing (3) and (4) we did not use assumption (A2). Hence, from these two expressions and $p_1 = f_1$, we have

$$\frac{p_2}{f_2} = \frac{1 + \frac{c_2}{f_2}(1 - t_p)}{1 + \frac{c_1}{f_1}(1 - t_p)}. \text{ From this ii) follows}$$

iii) $$c_1/f_1 < c_4/f_4 \Rightarrow p_1/f_1 = 1 < p_4/f_4$$

In establishing (3) and the right hand side of (6a) we did not use assumption (A2). From these two expressions and $p_1 = f_1$ we have

$$\frac{p_4}{f_4} \geq \frac{1 + \frac{c_4}{f_4}(1 - t_p)}{1 + \frac{c_1}{f_1}(1 - t_p)}. \text{ From this iii) follows}$$

Appendix 3

The results stated in section 4 are proved in this appendix. The results are proved by means of the Kuhn-Tucker conditions stated in appendix 1.

Proof of theorem 1: From (3) and (4) we have

$$\begin{Bmatrix} f_1 & c_1 \\ f_2 & c_2 \end{Bmatrix} \begin{Bmatrix} d_p \\ d_p^* \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}, \text{ where } d_p^* = d_p(1-t_p) \text{ for all p}$$

The above matrix has full rank since $c_1/f_1 > c_2/f_2$. Hence, we have that d_p and d_p^* are unique and independent of p. This means $d_{p'}^* = d(1-t_{p'}) = d(1-t_{p''}) = d_{p''}^* \Rightarrow t_{p'} = t_{p''}$ □

The key to this result is that the matrix has full rank.

Proof of theorem 2:

a) - b) From (1) it follows that $1/\hat{d}_v$ is independent of v. From (1) and (3) we have $1/\hat{d} = 1/\hat{d}_v = (f_i + c_i)/p_i \equiv 1 + r\ i = 1$, 3 and 4. $1/d$ and $1/d_v$ are interpreted as the private and corporate investors 1 + after tax return. From the definition of $\hat{d}$ og $\hat{d}_v$ we have

$$\hat{d} = \frac{1}{1+r} = \frac{(1-t_*)d}{1-t_*d} \Leftrightarrow (1-t_*)(1+r)d = 1 - t_*d \Leftrightarrow 1/d = 1 + r(1-t_*).$$

In the same way we have $1/d_v = 1 + r(1-t_v)$

c) From the expression for $1/d$, the definition of r, $p_1 = f_1$ and (4) we have

$$1/d = 1 + r(1-t_*) = 1 + \frac{c_1}{p_1}(1-t_*) = \frac{f_2}{p_2} + \frac{c_2}{p_2}(1-t_*)$$

From the above expression we have $\left(\frac{c_1}{p_1} - \frac{c_2}{p_2}\right)(1-t_*) = \frac{f_2}{p_2} - 1 > 0$ since $f_2 > p_2$

From the above expressions we have

$$1 + r = 1 + \frac{c_1}{p_1} = \frac{f_2 + c_2}{p_2} + t_*\left(\frac{c_1}{p_1} - \frac{c_2}{p_2}\right) \geq 1 + r_2,$$

where the inequality holds strictly for $t_* > 0$. Accordingly we have $r > r_2$ when $t_* > 0$.

d) - e) From $\hat{d}_v = \hat{d}$ and (1) for $i = 3$ and 4 we have that the right hand side of (5) and the left hand side of (6a) hold as equalities. Under the assumption $t_* > 0$ we will now prove, that the left hand side of (5) and the right hand side of (6a) hold as strict inequalities. Accordingly, from complementary-slack, private investors will not hold a short (long) position in black (orange) bonds.

It has previously been shown that $1/d = 1 + r(1 - t_*)$, where $r = (c_i + f_i - p_i)/p_i$ $i = 3$ and 4. By inserting in the left hand side (the right hand side) of (5) ((6a)) we have ($i = 3$ and 4).

$$[f_i + c_i(1 - t_*)]\, d = p_i \frac{f_i + c_i(1 - t_*)}{p_i + (c_i + f_i - p_i)(1 - t_*)} = p_i \frac{f_i + c_i(1 - t_*)}{f_i + c_i(1 - t_*) - t_*(f_i - p_i)}.$$

But this expression is larger than (less than) $p_3\,(p_4)$, since $f_3 > p_3\ (f_4 < p_4)$. □

Proof of theorem 3:

a) From the left hand side of (2a), (4) and the expressions for d_v and d we have

$$\frac{f_2 + c_2(1 - t_v)}{1 + r(1 - t_v)} \geq \frac{f_2 + c_2(1 - t_*)}{1 + r(1 - t_*)} \Leftrightarrow (c_2 - f_2 r)(t_* - t_v) \geq 0 \Leftrightarrow$$

$$f_2\left(\frac{c_2}{f_2} - \frac{c_1}{f_1}\right)(t_* - t_v) \geq 0 \Leftrightarrow (\text{since } r = c_1/f_1)$$

(7) $$t_* \leq t_v \text{ for all } v \quad \left(\text{follows from } \frac{c_2}{f_2} < \frac{c_1}{f_1}\right)$$

From the right hand side of (2a), (4) and the expressions for d and d_v we have

$$\frac{f_2 + c_2(1 - t_*)}{1 + r(1 - t_*)} \geq \frac{f_2 + c_2}{1 + r} \Leftrightarrow t_*(f_2 r - c_2) \geq 0 \Leftrightarrow$$

(8) $$t_* f_2\left(\frac{c_1}{f_1} - \frac{c_2}{f_2}\right) \geq 0 \Leftrightarrow t_* \geq 0$$

Accordingly we have $0 \leq t_* \leq \min\limits_v t_v$.

b) Suppose $t_* > 0$. Hence (8), and accordingly the right hand side of (2a), hold as strict inequalities. This implies, from complementary-slack, that the corporate investor will not hold a long position in bond no. 2.

Assume the existence of a company v' such that $t_{v'} > t_*$. Hence (7), and accordingly the left hand side of (2a), hold as strict inequalities. This implies, from complementary-slack, that the corporate investor v' will not be short in bond no. 2. □

Lemma 1: Absence of infinite arbitrage opportunities for all investors implies that a private investor's taxable income is reduced by buying an arbitrage portfolio composed of a long position in bond no. 2 and a short position in bond no. 1 or 4. The corporate investor's taxable income is increased by exactly the same amount by issuing the arbitrage portfolio.

Proof: The arbitrage portfolio is long in 1 unit of bond no. 2 and short in p_2/p_1 (p_2/p_4) units of bond no. 1 (4). We only analyze the case where bond no. 1 constitutes the short element.

Δ Taxable income for the private investor = $c_2 - \dfrac{p_2}{p_1} c_1 =$

$$c_2 - \frac{p_2}{p_1}(c_1 + f_1 - p_1) = p_2\left(\frac{c_2}{p_2} + 1 - \frac{c_1 + f_1}{p_1}\right) < p_2\left(\frac{c_2 + f_2}{p_2} - \frac{c_1 + f_1}{p_1}\right)$$

$= p_2(r_2 - r) \leq 0$, where we have utilized $p_1 = f_1$, $p_2 < f_2$ and $r_2 \leq r$.

Δ Taxable income for the corporate investor = $\dfrac{p_2}{p_1}(c_1 + f_1 - p_1) - c_2$

= $-\Delta$ Taxable income for the private investor □

Proof of theorem 4: Suppose that $\hat{t}_p > t_*$. Define $t_* = \min\limits_v t_v$, where $\min\limits_v t_v$ is the lowest corporate marginal tax rate after all arbitrage has been made. Suppose there exists a private investor such that $\underline{t}_p < t_*$. Hence, from the assumption $\hat{t}_p > t_*$ there also exists at least one private investor such that $\bar{t}_p > t_*$. These private investors can now, by means of bond no. 1 and 2, engage in mutual advantageous arbitrage trades. Concerning the first private investor, this arbitrage trade stops when $\underline{t}_p = t_*$, since $[f_1 + c_1(1 - t_*)]/p_1 = [f_2 + c_2(1 - t_*)]/p_2$. The arbitrage trades keep the total taxable income unchanged inside the group of private investors. From the above argument we have $t_p \geq t_*$ for all private investors. From the assumption $\hat{t}_p > t_*$ there exists at least one private investor where $t_p > t_*$. The high taxed private investor and the low taxed companies can now, again by means of bond no. 1 and 2, engage in mutual advantageous arbitrage trades. The arbitrage trades are profitable until we have $t_p = t_*$. Accordingly we have $t_p = t_*$ for all private investors. The second part of the theorem follows immediately from theorem 1 and 3. □

Appendix 4

Absence of infinite tax arbitrage opportunities for the corporate and private investors is equivalent to the statement that d_v and $d > 0$ satisfy

(1) - (2a) of appendix 1

(2b) $p_0 \geq [f_0 + c_0]\, \hat{d}_v$

(3) - (6a) of appendix 1

(6b) $[f_0 + c_0\,(1 - t_p)]\, d + \lambda_p \geq p_0 \geq [f_0 + c_0\,(1 - t_p)]\, d \quad \lambda_p \geq 0$

$$\boxed{c_0/f_0 > c_1/f_1 \Leftrightarrow p_0/f_0 > 1 = p_1/f_1}$$

From (1) (for $i = 1$), (2b) and $p_1 = f_1$ we have

$$\frac{p_0}{f_0} \geq \frac{1 + \frac{c_0}{f_0}}{1 + \frac{c_1}{f_1}}$$

From the above inequality it follows that $c_0/f_0 > c_1/f_1 \Rightarrow p_0/f_0 > 1$. In appendix 5 it is proved that there exists a corporate investor who increases his long position in bond no. 0. Hence, the above inequality holds as an equality (since (2b) holds as an equality). From this $p_0/f_0 > 1$ implies $c_0/f_0 > c_1/f_1$. □

Appendix 5

The statements described under item A, B and C in section 5 are proved in this appendix.

Since $t_p = t_*$ for all p, it suffices to analyze a single private investor. We define $\overline{x} = \sum_p \bar{x}_0^1\,(p)$. For this case to be an interesting case we assume $\overline{x} > 0$. Instead of solving the convex programs (MPP) and (MPV) we solve the corresponding convex programs (MPP1) and (MPV1), which are based on maximizing the after tax payments at time 1 subject to the condition that the portfolio generates non negative payments at time 0. (MPP1) and (MPV1) also give rise to the Kuhn-Tucker conditions (1) - (6b). In this appendix we will take (MPP1) and (MPV1) as the starting point. As (MPP1) is a convex program we have found an optimal solution to (MPP1), when this solution is feasible for (MPP1) and satisfies the Kuhn-Tucker conditions (3) - (6b). The same reasoning applies for (MPV1).

A. TI conditions a personal marginal tax rate of zero

Assertion: $x_0^s(p) = \overline{x}$, $x_1(p) = x_0^s(p)\, p_0/p_1$ and all remaining $x(p)$'s equal to zero are an optimal solution.
$x_0^1(v') = \overline{x}$, $x_1(v') = -x_0^1(v')\, p_0/p_1$ and all remaining $x(v')$'s equal to zero are an optimal solution.

Proof: The solution is feasible for (MPP1) and (MPV1) and clears the market. By assumption the solution imply $t_* = 0$. From this and (3) and (1) we have $\hat{d}_v = \hat{d} = d$. By this, (1) and (3) - (6a) are satisfied. From $x_0^s(p) = \sum_v x_0^1(v) = \overline{x}$ it follows that there exists a v' such that $x_0(v') > 0$. From the strictly positive quantities we have

$$(f_0 + c_0)\, d = p_0 - \lambda_p$$
$$(f_0 + c_0)\, \hat{d}_{v'} = p_0$$

Since $d = \hat{d}_{v'}$ we have $\lambda_p = 0$.

It is noticed that the right hand side of (2a) holds as an equality and the left hand side holds as a strict inequality. That is $x_2^s(v) = 0$. In this way we have found a feasible solution, which clears the market, and where (1) - (6b) are satisfied. □

From the above proof the stated (item A, section 5) clientele results follows immediately. It is noticed that there exists various optimal solutions (generally this applies to all three cases). More than one company could serve as counterpart to the private investors. Moreover, when $\overline{x}$ is ample the arbitrage trades could be reduced without hurting the profits.

B. TI conditions a personal marginal tax rate, which is larger than zero but less than the smallest corporate marginal tax rate

Assertion: The solution described in case A is also optimal in this case.

Proof: The solution is feasible for (MPP1) and (MPV1) and clears the market. By assumption the solution imply $0 < t_* < \min_v t_v$. We will first show that the stated solution imply $\lambda_p > 0$. Assume $\lambda_p = 0$. From (6b), (2b), (3) and (1) for $i = 1$ we have $t_* = 0$, which is a contradiction to the assumption $t_* > 0$. $\lambda_p > 0 \Rightarrow x_0^1(p) = 0$, which is in accordance with the

Kuhn-Tucker conditions. We will now show that (2a) holds as two strict inequalities. By utilizing (4) in the right hand side of (2a) combined with (1) and (3) we have

$$[f_2 + c_2(1-t_*)]\,d \ge (f_2+c_2)\,\hat{d}_v \;\Leftrightarrow\; \frac{f_2 + c_2(1-t_*)}{f_1 + c_1(1-t_*)} \ge \frac{f_2+c_2}{f_1+c_1} \;\Leftrightarrow$$
$$t_* f_1 f_2 \left[\frac{c_1}{f_1} - \frac{c_2}{f_2}\right] \ge 0.$$

But the last expression holds as a strict inequality. This means that the right hand side of (2a) also holds as a strict inequality (in accordance with $x_2^1(v) = 0$).

By inserting (4) in the left hand side of (2a) and henceforth utilizing (1), (3) and $p_1 = f_1$ we have (for all companies)

$$(*)\quad \frac{f_2 + c_2(1-t_v)}{f_1 + c_1(1-t_v)}\,f_1 \ge \frac{f_2 + c_2(1-t_*)}{f_1 + c_1(1-t_*)}\,f_1 \;\Leftrightarrow\; (t_* - t_v)\,f_1 f_2 \left[\frac{c_2}{f_2} - \frac{c_1}{f_1}\right] \ge 0.$$

This expression holds as a strict inequality since $t_* < \min\limits_v t_v$.

In this way we have that the proposed solution satisfies (1) - (6b). The remaining clientele effects are deduced from $t_* > 0$, (5) (6a) and (1) as these statements imply $x_3^s(p) = x_4^1(p) = 0$. □

Define $\underline{t}_v = \min\limits_v t_v$. From the above proofs we have $[f_0 + c_0(1-t_*)]\,d + \lambda_p = p_0$. By inserting d from (3) we have $\lambda_p(t_*) = p_0 - [(f_0 + c_0(1-t_*))/(f_1 + c_1(1-t_*))]\,p_1$. It is noticed that $\lambda_p(t_*)$ is a strictly increasing function of t_*. We have

$$\lambda(t_*)\,|_{t_*=0} = 0$$
$$\lambda(t_*)\,|_{t_*=\underline{t}_v} = p_0 - \frac{f_0 + c_0(1-\underline{t}_v)}{f_1 + c_1(1-\underline{t}_v)}\,p_1\ (>0)$$

Case A corresponds to $\lambda(t_*) = 0$

Case B corresponds to $\lambda(t_*) \in \,]\,0,\ \lambda(t_*)\,|_{t_* = \underline{t}_v}\,[$

Case C corresponds to $\lambda(t_*) = \lambda(t_*)\,|_{t_* = \underline{t}_v}$

C. TI conditions a personal marginal tax rate, which is larger than or equal to the smallest corporate marginal tax rate

Assertion: The solution stated below is an optimal solution

(**) $x_0^s(p) = \bar{x},\ x_1(p) = x_0^s(p)\, p_0/p_1 + \Delta x_1(p)$

(***) $x_2(p) = y,\ \Delta x_1(p) = -x_2(p)\, p_2/p_1$

All remaining $x(p)$'s are equal to zero, and $y \geq 0$ is chosen in such a way that $t_* = \min_v t_v$

The companies enter the opposite portfolios.

Proof: The solution is feasible for (MPP1) and (MPV1) and clears the market. We have to include the portfolio element (***), since (**) and assumption C imply $t_* \geq \min_v t_v$. If this inequality holds strictly we have from the above proof, that (*) is not satisfied. But then the left hand side of (2a) is not satisfied, for which reason the solution (**) alone is not optimal. If $t_* = \min_v t_v$ we have that $y = 0$ is optimal. Suppose $y = 0$ implies $t_* > \min_v t_v$. Now, we have to reestablish the left hand side of (2a). Since the tax functions are progressive we can choose $y > 0$ in such a way that $t_* = \min_v t_v$. The last statement implies that (2a) is reestablished. All y, where $t_* = \min_v t_v$, are optimal solutions. If y is chosen larger we have $t_* < \min_v t_v$, for which reason (*) is satisfied as a strict inequality. This implies $x_2^s(v) = 0$ for all v. But this statement is a contradiction to $x_2(p) = \sum_v x_2^s(v) = y > 0$. The clientele results from case B are modified by the statement that $x_2^s(v) \geq 0$. □

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Summary

On Black, Blue and Orange Bonds: A Tax Arbitrage Model with Asymmetric Taxation

This paper considers a one-period economy with both private and corporate investors. The market under consideration consists of three types of bonds (called black, blue and orange), which in one way or another differs with respect to taxation. It is shown that the absence of infinite tax arbitrage opportunities implies that all private investors will end up with the same marginal tax rate. This tax rate is no larger than the smallest corporate marginal tax rate. At the same time all types of bonds except claims selling at par exhibit clienteles. In the literature on bond arbitrage the clienteles are created by means of transaction costs or a (partial) ban on issuing bonds. The exhibition of clienteles in this model hinges on asymmetric taxation. Concluding the paper is a discussion of the effect of introducing previously issued blue bonds into the analysis.

Zusammenfassung

Über schwarze, blaue und orangefarbene Wertpapiere:[7] Ein Steuerarbitragemodell bei asymmetrischer Besteuerung

In diesem Beitrag wird ein auf eine einzige Wirtschaftsperiode bezogenes Modell für Anleger (sowohl Privatpersonen als auch Unternehmen) untersucht. Der untersuchte Markt umfaßt drei Arten von Wertpapieren, die hier schwarz, blau und orangefarben genannt werden und bei denen die Besteuerung auf die eine oder andere Weise unterschiedlich ist. Es wird gezeigt, daß das Fehlen von unbegrenzten Steuerarbitragemöglichkeiten impliziert, daß für alle privaten Anleger derselbe Grenzsteuersatz gilt. Dieser Steuersatz ist nicht höher als der niedrigste Körperschaftsgrenzsteuersatz. Gleichzeitig wird allen Arten von Wertpapieren mit Ausnahme von zum Nennwert verkauften Werten die Clientèle zugeordnet. In der Literatur über Wertpapierarbitrage ergibt sich die Clientèle jeweils aus den Transaktionskosten oder aus einem (teilweise verhängten) Emissionsverbot.

[7] Die Besteuerung der einzelnen Wertpapiere hängt von dem spezifischen Anleger ab, ob er Käufer oder Emittent ist, sowie von dem Typ von Anleger (Privatperson oder Unternehmen).

Die für dieses Modell gewählte Form der Darstellung der Clientèle hängt von asymmetrischer Besteuerung ab. Am Schluß dieses Beitrags findet sich eine Diskussion der Auswirkungen, die durch die Einbeziehung von zuvor ausgegebenen Standardwerten in die Analyse entstehen.

Résumé

Les obligations noires, bleues et oranges: Un modèle d'arbitrage fiscal avec une taxation asymétrique

Ce travail examine une économie d'une période avec des investisseurs individuels et des sociétés. Le marché considéré consiste en trois types d'obligations (appellées noires, bleues et oranges) qui ont une taxation différente. Il est montré que l'absence de possibilités d'arbitrage fiscal illimité implique que tous les investisseurs privés auront le même taux marginal d'imposition. Ce taux de taxation n'est pas plus élevé que le taux marginal de taxation le plus bas des sociétés. En même temps, tous les types d'obligations, à l'exception des celles vendues au pair, exhibent des clients. Dans la littérature sur l'arbitrage des obligations, la clientèle est créée au moyen de coûts de transaction ou d'un retrait partiel d'obligations émises. L'exhibition de la clientèle dans ce modèle dépend de la taxation asymétrique. En conclusion de l'article, on discute de l'effet d'une introduction dans l'analyse d'obligations bleues émises antérieurement.