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Hyperbolic Discounting Models in Prescriptive Theory of Intertemporal Choice*

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1. Introduction

The time preference literature today is divided into two different lines of research. The prescriptive theory presents standard discounted value as the only rational model of intertemporal choice. The standard discounting model applies a constant discount factor to each period. Maybe the strongest argument in favour of standard discounting has been brought forward by Strotz (1956), who argues that the only dynamically consistent model of intertemporal choice is the standard discounted value model. Koopmans (1960) has introduced the axiom of stationarity which forms the basis of the standard discounted value model. Today this model is still central in most prescriptive analyses of intertemporal choice (see, e.g., Chew and Epstein, 1990).

Prescriptive theory is contrasted by the line of descriptive research that shows how decision makers systematically deviate from behavior which can be modelled by standard discounted value. Experimental work in this field has been initiated by Thaler (1981) and advanced by works of, among others, Loewenstein (1987, 1988), Benzion et al. (1989), Loewenstein and Thaler (1989) and, more recently, Shelley (1993) and Loewenstein and Prelec (1993). Many deviations from standard discounting have been found. For instance, people do not apply a constant discount factor to each period but use discount factors that decrease over time. Thus delaying an outcome from period 0 to period 1 is worse than delaying it from period 10 to period 11. Also, in a sequential context, a sequence's value is not the sum of the values of its parts, as any time-additive model such as standard discounting predicts. The attractivity of an outcome sequence increases if the periods' outcomes are uniformly spread or if they improve over time.

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A model which can describe this sequence effect has been proposed by Loewenstein and Prelec (1993). Loewenstein and Prelec define a value of the improvement trend of the sequence and a value of the uniformity of the sequence. These may be negative if the sequence deteriorates over time or has a high variation. The overall value of the sequence is then defined to be sum of the values of its component parts plus the improvement and uniformity values. A similar idea can also be found in Eisenführ (1988). These models are not axiomatized[1] and are therefore purely descriptive.

Unlike these models, theories which describe decreasing discount factors over time, the so called hyperbolic discounting models (Ainslie and Haslam, 1992, Loewenstein and Prelec, 1992) have been axiomatized (Harvey, 1986). Hyperbolic models were first used in behavioral theory of animals (Chung and Herrnstein, 1967, Ainslie, 1975), where it was shown that the effectiveness of a reward is inversely proportional to reward delay. In terms of discounting, this means that the discount factor applied to each period declines as a (hyperbolic) function of time. The intuition behind this approach is that decision makers are the more sensitive to timing changes the closer these changes are to the present.

While we agree that due to the lack of axiomatization, models incorporating sequence effects cannot be prescriptively justified, the purpose of this paper is to show that hyperbolic models can be used in prescriptive theory. Hyperbolic discounting models have been proposed as a basis for prescriptive theory (Harvey, 1986 and 1992, Eisenführ and Weber, 1994, chapter 12). Most of the time preference literature today, however, presents hyperbolic models as purely descriptive theories (see, e.g., Ainslie, 1991, Rachlin and Raineri, 1992).

This is partly because the axiomatic basis of hyperbolic models is not seen as convincing as the axiomatic basis of the standard model. We will argue that this is not true. In section 2 we will present an axiomatically based general theory of intertemporal choice. We will distinguish between ordinal (subsection 2.1) and cardinal (subsection 2.2) approaches to the theory. In subsection 2.3 we will relate period decision weights to discount factors by writing $w_t = 1/(1+i)^{\alpha(t)}$ where $\alpha(t)$ is a time perception function. The standard discounting model corresponds to a linear α (subsection 3.1) whereas hyperbolic models refer to nonlinear αs (subsection 3.3). Thus just like standard discounting, hyper-

[1] For an axiomatized model that can accomodate preference for variation, see Gilboa (1989), who weakens the independence axiom that leads to the additive model. A framework for a possible axiomatization of preference for trend and variation is given in Dyckhoff (1988, Table 2, p. 996), for the case that one postulates interval scale invariance.

bolic discount functions are not arbitrary but are founded on sound principles of rationality.

Standard discounted value's and hyperbolic models' axiomatizations differ only with regard to two axioms whose usefulness for prescriptive purposes needs to be discussed. We argue in subsection 3.2 that except for the case that the decision to be taken will be reevaluated, i.e. the original decision and the reevaluation later need to be dynamically consistent, neither axiom can be considered superior.

The case of regular future reevaluations of today's decision was first considered by Strotz (1956). He showed that only standard discounting is dynamically consistent. Dynamic inconsistency is the second reason why hyperbolic discounting models have been rejected. However, only *if* today's decision will be reevaluated in the future and, upon reevaluation, possibly be changed, is dynamic consistency an issue. It needs to be noted that many decisions are binding for the future: in management, such are decisions on mergers or acquisitions, the building of new production facilities or the hiring of personnel. In personal decision making, decisions that are binding for the future are career choice or renting or buying an apartment or house. For such decisions the question of dynamic consistency is irrelevant.

Which model to use should be decided by considering which underlying axiom better *describes* the decision maker's time preference. We show in subsection 3.4 that experimental evidence clearly identifies the axiom leading to hyperbolic discounting as the better description of real time preference. Section 4 concludes with our main message: when a decision maker whose time preference is correctly described by hyperbolic models wants to rationally take a decision that will be binding for the future, he should not let standard discounted value models be forced upon him, but use hyperbolic discounting models in accordance with his personal time preference.

Proofs of all theorems, or references where proofs can be found elsewhere in the literature, are relegated to the appendix.

2. General theory of time preference

2.1 Ordinal preferences

Formally, intertemporal choice can be viewed as a multi-attribute decision problem. We will write $a = (a_0, a_1, \ldots, a_{T(a)}) \succ b = (b_0, b_1, \ldots, b_{T(b)})$ if the decision maker prefers consequence stream a over b. The notation $\preceq$, $\prec$, $\succeq$ and $\sim$ is used in the obvious way. Here, $a_t \in X$ is an element of the set of possible consequences X and denotes the t^{th} period's consequence of stream a.

In writing $a_t \in X$ we assume that the set of possible consequences is the same for all periods. Most contributions to the time preference literature make this assumption although, clearly, there are decision problems where different types of consequences occur in different periods. We wish to analyze, e.g., how decisions will be affected if consequences are delayed. Delays from one period to the other only make sense if the sets of consequences that can occur in the two periods in question are the same.

The consequences we consider need not be monetary, nor numerically described at all. The consequence a_t may denote the number of kindergartens built each year or the annual consumption of a nonregenerating natural resource like oil or gas, or it may be a verbal description of an individual's state of health.

We want to allow the decision maker not only to state preferences between consequence *streams* but also between consequences in *specific periods*. To this end, we assume that there exists a neutral consequence $n \in X$. For the choice between monetary consequences it would make sense to let n = DM 0. Then, we can identify the consequence 'DM 1000 today' with the sequence (DM 1000, $n, \ldots$). By filling 'empty' periods with the neutral consequence n = DM 0, single consequences become consequence streams, between which there is a well-defined preference. The decision context will need to define what is to be understood by the neutral consequence. When period consequences are monetary, n might be DM 0. When they verbally describe one's state of health, n might be the description of one's current state of health.

A restriction we make, however, is that we only consider consequence streams a having sure consequences and a finite planning horizon $T(a) \in N_0$, which may depend on a. Thus we consider the set of alternatives $\mathcal{X} = \bigcup_{m \in N_0} X^m$. We make this finite horizon assumption because we want to avoid technical difficulties[2] arising from infinitely many attributes.

We wish to specify an intertemporal value function that describes the decision maker's preferences.

2 We will numerically represent preferences by value functions where the value of a consequence stream will be the sum of the values of its parts. Having infinitely many summands, the value of an infinite stream may be infinite, leaving it incomparable to other consequence streams which may also have infinite value.

Definition 1 (Ordinal intertemporal value function)

An ordinal intertemporal value function is a function $V : \mathcal{X} \to R$ *such that for all* $a, b \in \mathcal{X}$, $a \succeq b \Leftrightarrow V(a) \geq V(b)$.

First we need to discuss under what conditions an intertemporal value function exists. For this, we will need a topological assumption[3] which can, for most practical purposes, be considered to be true.[4]

Theorem 1 (Debreu, 1954)

Suppose that the topological assumption holds. Then a continuous ordinal intertemporal value function V exists if and only if the decision maker's preference is complete[5], transitive[6] and continuous[7]. Another intertemporal value function V′ describes the same ordinal preferences if and only if there exists a strictly increasing transformation $f : R \to R$ *such that for all* $a \in \mathcal{X}$, $V'(a) = f(V(a))$.

Completeness of the decision maker's preference demands that he be able to compare any two alternatives. Transitivity demands that, in our introductory example, if the decision maker prefers no withdrawal symptoms to light symptoms, and light symptoms to strong ones, then he prefer no symptoms to strong ones. Neither completeness nor transitivity should be controversial axioms of rationality. The continuity of the decision maker's preference is always assured if X is finite; if X is an interval of the real numbers, however, lexicographic preferences constitute an example of noncontinuous preferences.

The mere existence of an intertemporal value function is not enough for a decision analyst. In order to apply it, this value function needs to be of a tractable functional form. The following axiom will ensure that V takes the simple additive form.

In order to define this axiom, we need some notation. Let $I \subset N_0$, $\bar{I}$ be its complement and let $a, y \in \mathcal{X}$ be two consequence streams. Then $a_I y_{\bar{I}} \in \mathcal{X}$ denotes the consequence stream with t^{th} period's consequence a_t if $t \in I$ and y_t if $t \in \bar{I}$. If I contains only a single element t, we will write a_t instead of $a_{\{t\}}$. For instance, if a denotes the constant stream of

3 *Topological assumption:* X is a compact topological space. $\mathcal{X}$, endowed with the product topology, is connected and separable. For the definitions of a topology, compactness, connectedness and separability see Kelley (1955), or any other standard textbook on topology.

4 If X is an interval of the real numbers (endowed with the Euclidean topology), or if X is finite (endowed with the topology generated by all sets $\{y \in \mathcal{X} \mid y \succ x\}$ and $\{y \in \mathcal{X} \mid x \succ y\}$), the topological assumption is always satisfied.

5 i.e., for all $a, b \in \mathcal{X}$, either $a \succeq b$ or $b \succeq a$.

6 i.e., for all $a, b, c \in \mathcal{X}$, if $a \succeq b$ and $b \succeq c$, then $a \succeq c$.

7 i.e., for all $x \in \mathcal{X}$, the sets $\{y \in \mathcal{X} \mid y \succ x\}$ and $\{y \in \mathcal{X} \mid x \succ y\}$ are open in $\mathcal{X}$.

an annuity of DM 1000, and y the constant stream of an annuity of DM 5000, $a_0 y_{\bar{0}}$ = (DM 1000, DM 5000, ..., DM 5000).

Axiom 1 (Mutual preference independence)

For all $I \subset N_0$ *and all* $a, b, y, z \in \mathcal{X}$, $a_I y_{\bar{I}} \succeq b_I y_{\bar{I}} \Leftrightarrow a_I z_{\bar{I}} \succeq b_I z_{\bar{I}}$.

The intuition behind this axiom is as follows. The first two alternatives have in common all consequences in periods belonging to $\bar{I}$, namely all consequences that refer to $y_{\bar{I}}$. Then Axiom 1 demands that the $y_{\bar{I}}$ part of both streams be irrelevant to the decision maker. The preference statement $\succeq$ should therefore only stem from comparing the different consequences in the periods belonging to I, i.e. the a_I versus the b_I part. By the same argument, the $z_{\bar{I}}$ part of the second two alternatives should be irrelevant to the second preference statement. Thus the second preference statement, too, should only stem from a comparison of the a_I and b_I parts. Then, clearly, both preference statements are the same.

This axiom is a clear requirement of rationality. Identical components (the $y_{\bar{I}}$ and $z_{\bar{I}}$ parts) should not influence the choice between two alternatives. If they did, the individual would not be able to state a preference between one period's consequences, disregarding the other periods. In our introductory example, however, the individual surely would, and should, be able to state that he preferred to be healthy rather than ill today, regardless of what his future state of health would be.

As a consequence of their rational appeal, independence assumptions are key to all fields of decision theory. An interesting discussion of the history of the independence assumption can be found in Fishburn and Wakker (1995).

As already stated, Axiom 1 forces V to take on the additive form:

Theorem 2 (Debreu, 1960)

Suppose that the assumptions of Theorem 1 hold and that all periods are essential.[8] *Then Axiom 1 holds if and only if there exist* $w_t \in R$, $w_t > 0$, *and functions* $v_t : X \to R$ *such that*

$$V(a_0, \ldots, a_{T(a)}) = \sum_{t=0}^{T(a)} w_t \, v_t(a_t) \tag{1}$$

is a continuous ordinal intertemporal value function. Another intertemporal value function

[8] i.e., for all $t \in N_0$ there exist $a, b \in X$ such that $a_t \succ b_t$. The only purpose of this trivial assumption is to exclude the possibility that some $w_t = 0$.

$$V'(a_0, \ldots, a_{T(a)}) = \sum_{t=0}^{T(a)} w'_t \, v'_t(a_t),$$

describes the same ordinal preferences as does V if and only if for all $t \in N_0$ *there exist* k_t, $l \in R$, $l > 0$, *such that* $w'_t v'(a_t) = l w_t v(a_t) + k_t$.

The v_ts are period value functions for consequences in period t which can be normalized such that $v_t(n) = 0$ and $\max_{a \in X} v_t(a) - \min_{a \in X} v_t(a) = 1$. The first of these normalizations is to express the idea that the consequence n is neutral. The second normalization makes sure that all v_ts have the same range, namely an interval of length one. If $v_t(n) = 0$ denotes the worst possible period consequence, we can take this interval to be [0,1]. The constants w_t are the decision weights of the periods, which we can normalize such that $w_0 = 1$.

It is important to note the implications of Axiom 1. It excludes the possibility of modelling sequence effects, that is preference for trend or smoothness. Loewenstein and Sicherman (1991), e.g., have shown that indivuals generally prefer increasing wage profiles to decreasing ones. The annual increase motivates them for their jobs since it is seen as a sign of improving work efficiency. If we wanted to model preference for improving sequences, the value which a_t adds to the overall value of a would depend on whether a_t is better than a_{t-1}. In Theorem 2, however, $v_t(a_t)$ is independent of consequences in period $t - 1$.

So far, we have introduced an intertemporal value function which, through its simple additive form, is a suitable basis for analysis of intertemporal choice. We will, however, still need to simplify V for our purposes. What limits the tractability of (1) most is the fact that different value functions v_t may have to be applied to different periods. Formula (1) would allow that the decision maker preferred more money to less today (if v_0 was increasing) and, at the same time, less money to more tomorrow (if v_1 was decreasing). Since the same type of consequences are evaluated in each period, be they numerically described, possibly monetary, or verbally described states of health, it would not make much sense to apply different value functions to different periods. We will now present and compare two different axiomatic approaches that ensure that we can take the same v_t for all periods, that is, that imply that

$$V(a_0, \ldots, a_{T(a)}) = \sum_{t=0}^{T(a)} w_t \, v(a_t). \tag{2}$$

Note that (2) requires the consequence sets for all periods to be identical, since the same period value function v is used for evaluation in all periods.

The first approach goes back to Dyckhoff (1988, p. 1002). Suppose you are endowed with constant income (consequence) stream $a = (a_0, \ldots, a_{T(a)})$ with $a_0 = \cdots = a_{T(a)} = \text{DM } 1000 =: x$ and you apply (1) to evaluate substitutions between periods. Suppose that all v_ts are differentiable. You wish to specify how many additional DM you would require in period $t + 1$ in order to exactly compensate a loss of DM 1 (or, an infinitesimally small loss) in period t. The answer is given by the marginal rate of substitution $MRS_{t,t+1}(x)$ between periods t and $t + 1$ at $x = \text{DM } 1000$, which is

$$MRS_{t,t+1}(x) = \frac{w_t}{w_{t+1}} \frac{v_t'(x)}{v_{t+1}'(x)}.$$

Thus, $MRS_{t,t+1}(x)$ depends on two factors: first, it depends on the decision weights of periods t and $t + 1$. The higher the relative weight of period t as compared to the weight of period $t + 1$, i.e. the higher w_t / w_{t+1}, the more DM you require in $t + 1$ for compensation. Second, $MRS_{t,t+1}(x)$ also depends on the additional values $v'(x)$ an extra DM induces in periods t and $t + 1$. The more the 1001^{st} DM is valued in period t as compared to period $t + 1$, i.e. the higher $v_t'(x)/v_{t+1}'(x)$, the more DM you require in $t + 1$ for compensation.

Dyckhoff (1988, p. 1002 and p. 1007) was able to show that if the marginal rate of substitution is independent of the derivatives of the v_ts, then all v_ts must be identical. We will first formally state his axiom and his theorem and then discuss it.

Axiom 2 (Constant Marginal Rates of Intertemporal Substitution)

For all $t \in N_0$ and all $x \in X$, $MRS_{t,t+1}(x) = w_t / w_{t+1}$.

Theorem 3 (Dyckhoff, 1988)

Suppose that the assumptions of Theorem 2 hold and that all v_ts are differentiable. Then (2) is an ordinal intertemporal value function if and only if Axiom 2 holds.

We see that Axiom 2 does ensure, as has been our objective, that all v_ts are equal such that we may drop the subscript t for the period value functions. What is missing is an intuitive interpretation of Axiom 2 as a requirement of rationality, as has above been given for the axiom of mutual preference independence. Neither are we able to give Axiom 2 an intuitive meaning nor does Dyckhoff himself present any justification. It therefore stands as no more and no less than 'the right formula to do the job'. Whenever X is finite, marginal rates of substitution cannot be defined since then the v_ts are not differentiable. Thus for a finite X, Axiom 2 cannot be applied.

We will now present another approach that also achieves equality of the v_ts and that can be given an intuitive meaning. Doing so, however, requires us to assume that the decision maker can compare strenghts of preferences. We need to first develop the theory of cardinal preferences.

2.2 Cardinal preferences

The strength of preference of stream a over stream b is interpreted as the value the decision maker attributes to a change from b to a. It will be denoted as $b \to a$. The preference statement $b \to a \succ d \to c$ means that a is more strongly preferred to b than is c to d. Or, in other words, the 'preference difference' between a and b is perceived to be greater than the one between c and d. For simplicity, we use the same notation $\succ, \prec, \succeq, \preceq, \sim$ for the strength of preference relation on $\mathcal{X} \times \mathcal{X}$ as for preference relation on $\mathcal{X}$ between consequences.

Definition 2 (Cardinal intertemporal value function)

A cardinal intertemporal value function is an ordinal intertemporal value function with the additional property that for all $a, b, c, d \in \mathcal{X}$, $b \to a \succeq d \to c \Leftrightarrow V(a) - V(b) \geq V(c) - V(d)$.

This is to say that the value difference between two alternatives measures the strength of preference between these alternatives. Making use of the strength of preference statements is common in prescriptive decision analysis. The work by Dyer and Sarin (1979) is central to most procedures of weight elicitation in a multi-attribute decision context.

The question whether a strength of preference relation is a meaningful concept has been a controversial subject in economics. It has been argued that the strength of preference relation cannot be derived from real choices. Wakker (1989, p. 35), however, has developed a theory that explains how cardinal preferences can be inferred from revealed ordinal choices. Since cardinal preferences are a well-defined concept, we will now identify conditions under which a cardinal intertemporal value function exists. We will then compare our approach with Wakker's key idea.

Under the topological assumption, existence of an ordinal intertemporal value function V_{ord} and a function $V_{card} : \mathcal{X} \to R$ such that $V_{card}(b \to a) \geq V_{card}(d \to c)$ whenever $b \to a \succeq d \to c$ can be guaranteed by Theorem 1. The problem is to find conditions under which V_{card} really is cardinal in the sense of Definition 2. For this, V_{ord} and V_{card}

need to be compatible, i.e., $V_{card}(b \to a) = V_{ord}(a) - V_{ord}(b)$. The following theorem states which consistency conditions are needed:

Theorem 4

Suppose the topological assumption holds. Then a continuous cardinal intertemporal value function V exists if and only if both the ordinal and cardinal preferences are complete, transitive and continuous and for all $a, b, c, d \in \mathcal{X}$,

(i) $b \to a \succeq d \to c \;\Leftrightarrow\; a \to b \preceq c \to d.$

(ii) $b \to a \succeq d \to c \;\Leftrightarrow\; c \to a \succeq d \to b.$

(iii) $b \to a \succeq c \to c \;\Leftrightarrow\; a \succeq b.$

Another intertemporal value function V' *describes the same cardinal preferences as does V if and only if there exist* $k, l \in R$, $l > 0$, *such that for all* $a \in \mathcal{X}$, $V'(a) = l\,V(a) + k$.

The consistency conditions are intuitive. As for (a), note that if $b \to a$ is considered an improvement, then, necessarily, the converse exchange $a \to b$ needs to be considered as a deterioration. Condition (a) demands that the more preferred the improvement of an exchange, the less preferred the converse exchange (and vice versa). Condition (b) is a necessary consequence, since $V(a) - V(b) \geq V(c) - V(d)$ if and only if $V(a) - V(c) \geq V(b) - V(d)$. As for (c), note that $c \to c$ is a 'null exchange' which should never be preferred to a change from the less preferred alternative b to the more preferred a. It should not be controversial to accept these consistency conditions as rationality requirements. Thus by Theorem 4, the existence of a well-defined cardinal intertemporal value function can safely be assumed.

We require more than the mere existence of a cardinal intertemporal value function V. In order to derive the additive form (1), consider the following axiom.

Axiom 3 (Mutual difference independence)

For all subsets $I \subset N_0$ *and all* $a, b, y, z \in \mathcal{X}$, $b_I y_{\bar{I}} \to a_I y_{\bar{I}} \sim b_I z_{\bar{I}} \to a_I z_{\bar{I}}$.

The intuition is as for the definition of preference independence: Axiom 3 requires the $y_{\bar{I}}$ and $z_{\bar{I}}$ parts to be irrelevant for assessing the preference difference, leaving both of them to depend on the a_I and b_I consequences only. Then, both preference differences are the same.

Having the same intuitive interpretation as the axiom of mutual preference independence, both the axiom of mutual preference – and difference – independence have the same appeal for a prescriptive analysis. If mutual difference independence failed to hold, the decision maker would not be able to make strength of preference statements concerning one period's consequences disregarding other periods. This would severely limit the tractability of any analysis by requiring to constantly refer to all periods whenever making a cardinal preference statement. Therefore, Axiom 3 should be assumed for prescriptive theory.

We have stated Axiom 3 in a more general form than would have been needed: as Dyer and Sarin (1979) have shown, one only requires difference independence of one period from all others for a cardinal additive multi-attribute value function. The following theorem states the consequence of assuming Axiom 3:

Theorem 5 (Dyer and Sarin, 1979)

Suppose that the assumptions of Theorems 2 and 4 hold. Then (1) is a cardinal intertemporal value function if and only if Axiom 3 holds.

We note that other approaches to cardinal theory are possible. Krantz et al. (1971) or French (1988) develop the theory replacing topological for algebraic solvability assumptions. Wakker (1989) does not assume cardinal preferences as primitives but derives cardinal from ordinal preference statements[9]. The distinct advantage of Wakker's approach is that he needs only ordinal preferences as primitive while we need both ordinal and cardinal preferences as primitives. Unlike Wakker, we thus need a consistency requirement between ordinal and cardinal preferences. A disadvantage of Wakker's approach may be that he can only define cardinal preferences for period consequences, which need not necessarily be complete. We assume complete cardinal preferences between consequence streams, but we need the additional assumption of mutual difference independence to also be able to speak of cardinal preferences for period consequences.

We are now in a position to formulate under which circumstances all v_ts coincide. This axiom can be intuitively understood and can serve as an alternative to Axiom 2. It postulates that the way preference differences are compared is the same over all periods.

[9] For $a_t \succ b_t$ and $c_t \succ d_t$ Wakker (1989, p. 35) defines $b_t \to a_t \succ d_t \to c_t$ to mean that there exist $x, y \in \mathcal{X}$ such that (i) $(x_0, \ldots, x_{t-1}, c_t, x_{t+1}, \ldots, x_{T(x)}) = x_{\bar{t}}\, c_t \preceq y_{\bar{t}}\, d_t$ and (ii) $x_{\bar{t}}\, a_t \succeq y_{\bar{t}}\, b_t$. He interprets (i) to mean that the positive argument to obtain c_t instead of d_t is outweighed by the (apparently) negative argument to obtain $x_{\bar{t}}$ instead of $y_{\bar{t}}$. Similarly, (ii) can be interpreted to mean that the positive argument to obtain a_t instead of b_t outweighs the (as we know by (i)) negative argument to obtain $x_{\bar{t}}$ instead of $y_{\bar{t}}$. Then, via the 'measuring rod' provided by $x_{\bar{t}}$ and $y_{\bar{t}}$, we can conclude that the positive argument to obtain a_t instead of b_t weighs more heavily than does the positive argument to obtain c_t instead of d_t. Wakker then shows that the 'cardinal' preferences derived in this way are indeed described by the differences $v_t(a_t) - v_t(b_t)$ (Lemma II. 4.5., p. 37).

Axiom 4 (Constant preference differences)

For all $s, t \in N_0$ *and all* $a, b, c, d \in X$, $b_t \rightarrow a_t \succeq d_t \rightarrow c_t \Leftrightarrow b_s \rightarrow a_s \succeq d_s \rightarrow c_s$.

The notation a_t needs to be clarified. So far, we have used it to mean the t^{th} period's consequence of consequence *stream* a. Here, a does not denote a consequence stream but an element of X, i.e. a consequence in a specific period. For simplicity of the presentation, a_t is used as a shorthand for the consequence stream $a_t n_{\bar{t}}$ which has neutral consequences n in all periods but in period t, where it has consequence a.

In order to intuitively understand the axiom, suppose $b_t \rightarrow a_t \succ d_t \rightarrow c_t$. For instance, in period t, the preference difference between $a =$ DM 1001 and $b =$ DM 1000 is compared to the difference between $c =$ DM 1 and $d =$ DM 0. Suppose decreasing marginal value, i.e., the 1001^{st} DM is valued less than the 1^{st} one. Would there be any reason to believe that had these amounts not referred to period t, but to another period s, the 1001^{st} DM might have been valued *more* than the 1^{st} one? We could not imagine why it should. When comparing the 1001^{st} with the 1^{st} DM, the decision maker may not have considered period t at all – and it is our position that he should not have. Value difference judgments do not relate to time preference issues and should therefore be independent of the timing of the consequences. This is precisely what the axiom of constant preference differences demands of the decision maker. A similar axiom can be found in Krantz et al. (1971, Theorem 15, p. 305) or in Wakker (1986, Definition 4.1., p. 318).

We will assume constant preference differences as a rationality requirement. With it, we have the following

Theorem 6

Suppose the assumptions of Theorem 5 hold. Then (2) is a cardinal intertemporal value function if and only if Axiom 4 holds.

We see that the cardinal Axiom 4 of constant preference differences achieves identical v_ts just as does the ordinal Axiom 2 of constant marginal rates of substitutions.

2.3 Relating decision weights to discount rates

We have seen that both Theorem 3 and Theorem 6 lead to the same additive functional form (2) of an (ordinal or cardinal) intertemporal value function. From here on, we will leave it open whether (2) has been derived as an ordinal or cardinal intertemporal value function. Since we wish to interpret (2) as a discounting model, it is convenient to transform the decision weights w_t in order to examine their relation to discount rates. We will write

$$w_t = \frac{1}{(1+i)^{\alpha(t)}}.$$

Note that since $w_t \neq 0$, we can always find $\alpha(t)$, $i \in R$, which satisfy this equation. The normalization $w_0 = 1$ translates into $\alpha(0) = 0$, and $w_t > 0$ translates into $i > -1$. We have now transformed (2) into

$$V(a_0, \dots, a_{T(a)}) = \sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+i)^{\alpha(t)}}. \tag{3}$$

Here, time preference is described by a constant discount rate $1 + i$. The discounting of a future value $v(a_t)$ depends on how far t is perceived to lie in the future: the perception of t is described by $\alpha(t)$. An individual may perceive periods 101 and 102 to be closer to each other than periods 1 and 2, although the time difference is 1 period in both cases. A discounted value functional based on this individual's time preference will then have to discount more heavily between periods 1 and 2 than between 101 and 102: this can be accounted for via the time perception function $\alpha(t)$.

Neither $1 + i$ nor α alone determine the individual's intertemporal value function, only both these parameters together do. One could interpret $1 + i$ as the discount factor that is applied to one time unit (period), while α can be interpreted to describe how fast time is perceived to pass in the decision maker's mind. The idea that time preferences can be interpreted by saying that a decision maker's inner 'clock' runs at a different speed than does physical time can also be found in Loewenstein and Prelec (1992, p. 126). An increase in the discount per period can be compensated for if one time unit is perceived to last longer. This corresponds to lower $\alpha(t)$s. For instance, $i = 0.1$ and $\alpha(t) = t$ lead to the same intertemporal value function (3) as do $i = 0.21$ and $\alpha = t/2$ since $1.21^{t/2} = 1.1^t$. The following theorem formalizes this idea. It states that α is only determined up to positive linear transformations. This will also show that in particular, any linear α is equivalent to $\alpha(t) = t$, that is, to the standard discounting model.

Theorem 7

Suppose that (3) is an intertemporal value function. Another intertemporal value function

$$V'(a_0, \dots, a_{T(a)}) = \sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+j)^{\alpha'(t)}}$$

with $j \in R$, $\alpha : R \to R$ describes the same ordinal preferences as does V if and only if there exist $k, l \in R$, $l > 0$, such that $(1 + j)^l = 1 + i$ and for all $t \in N_0$, $\alpha'(t) = l \cdot \alpha(t) + k$.

In this theorem, the constant l corresponds to simultaneous, mutually compensating changes in the two parameters $1 + i$ and α, like in the simple example above. The constant k leads to a division of V by $(1 + j)^k$, that is, a positive linear transformation which leaves the properties of V as a cardinal intertemporal value function invariant.

We will allow different functional forms of $\alpha(t)$. The only restriction we will make is that periods that lie further in the future are, of course, also perceived to be temporally more remote: $t \geq t' \Leftrightarrow \alpha(t) \geq \alpha(t')$. We assume α to be an increasing function. We now want to define what is to be understood by patience or impatience. Suppose the series of the decision weights $(w_0, w_1, \ldots)$ is decreasing. This can be interpreted to mean that the decision maker wishes to have positive consequences rather earlier and negative consequences rather later, which is what is intuitively understood by impatience. Formally,

Definition 3 (Patience and Impatience)

The decision maker is said to be

(i) impatient if $(w_0, w_1, \ldots)$ is decreasing,

(ii) timing indifferent if $(w_0, w_1, \ldots)$ is constant, and

(iii) patient if $(w_0, w_1, \ldots)$ is increasing.

Of course, the series of the w_t may first decrease and then increase, or change direction several times. We restrict attention to the three easiest cases. We have the following

Theorem 8

Suppose that (3) is an intertemporal value function. Suppose further that $t \geq t' \Leftrightarrow \alpha(t) \geq \alpha(t')$. Then the decision maker is

(i) impatient if and only if $i > 0$,

(ii) timing indifferent if and only if $i = 0$, and

(iii) patient if and only if $i < 0$.

Note that under the assumption of an increasing α, there can be only the three patterns of definition 3.

We have now developed a theory of time preference, relying on axioms which can be intuitively understood and which offer themselves as conditions of rationality. Any rational decision maker should base his decision on the model developed so far. In the literature, further axioms have been discussed which imply special functional forms of the period decision weights. Since different axioms lead to different specifications of the decision weights, the resulting theories are no longer general. They do not apply to any rational decision maker, but only to those individual decision makers who accept the respective axioms. One of these axioms, stationarity, has been regarded as being a superior definition of rationality than other possible axioms. We will argue in subsection 3.2 that this is true only in a special decision context, namely if the decision is to be regularly subjected to revisions. The following section will present possible individual specifications of discounting models.

3. Individual discounting models

In this section, we identify how different forms of the time perception function lead to different discounting models. In subsection 3.1, we present standard discounted value. It is the simplest subclass of our general model since it corresponds to a linear α. We compare our axioms with the traditional axiomatic basis of standard discounted value, the stationarity assumption. In subsection 3.2., we discuss in which cases stationarity can, through dynamic consistency arguments, be interpreted as a requirement of rationality. In subsection 3.3, we turn to hyperbolic discounting models. We show that concave αs lead to hyperbolic models. In subsection 3.4, we compare both standard and hyperbolic discounting models with empirical data.

3.1 Linear time perception

Linear time perception leads to the standard discounted value model which has been axiomatized by Koopmans (1960). The key axiom is the stationarity assumption:

Axiom 5 (Stationarity)

For all $t, s, l \in N_0$ *and all* $a, b \in X$, $a_t \succeq b_s \;\Leftrightarrow\; a_{t+l} \succeq b_{s+l}$.

If a decision maker prefers DM 1100 in period 11 to DM 1000 in period 10, then stationarity demands that he prefer DM 1100 in period 1 to DM 1000 in period 0. Note that both preference statements are made *today*, in period 0. This is because, in the theory developed so far, we have defined present preferences between consequence streams only. If we say 'in period 10', this does *not* mean that in period 10, he *will* prefer immediate DM 1000 to DM 1100 delayed one further period. Rather, 'in period 10' refers only to the timing of the DM 1000, not to the timing of the preference statements. We will elaborate further on this point when we discuss whether or not Axiom 5 is a requirement of rationality.

Stationarity demands that the trade-off between two periods depend only on the temporal distance of the periods, which, here, is $t - s$ for periods t and s as well as for $t + l$ and $s + l$. Since the perception of the temporal distance of two consecutive periods is constant, a constant discount factor $1 + i$ needs to be applied between two consecutive periods, regardless of whether they are close to the present or remote in the future. This corresponds to a linear α, and thus, by Theorem 7, to $\alpha(t) = t$.

Theorem 9 (Koopmans, 1960)

Suppose that (3) is an intertemporal value function. Then Axiom 5 holds if and only if

$$V(a_0, \ldots, a_{T(a)}) = \sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+i)^t}. \tag{4}$$

Stationarity, in fact, implies a bit more than what is stated here. Theorem 9 assumes identical period value functions. We could as well have started with the more general intertemporal value function (1) instead of (3). Then, stationarity would still have implied that V takes on the functional form (4). That is, stationarity implies *both* identical period value functions *and* period decision weights of the form $w_t = \frac{1}{(1+i)^t}$. In this sense, it is a stronger assumption than are Axioms 2 or 4 which only imply identical period value functions. Since we intend to develop the theory step by step, we have introduced these axioms independently of stationarity. The decision maker may want to use identical period value functions, but we will argue in the next subsection that he will not necessarily want to use the specific period decision weights implied by stationarity.

Figure 1 shows the linear and different concave time perception functions α over 15 periods. We have normalized all αs such that $\alpha(5)$ is the same for all models. The other functions in Figure 1 will be explained in subsection 3.3.

3.2 Dynamic consistency and the rationality of stationarity

Is stationarity a requirement of rationality? We will argue that the answer to this question depends on the decision context. We have already noted in the introduction that supporters of standard discounting models build on Strotz (1956) who showed that it is the only dynamically consistent model of intertemporal choice. Consequently, when sta-

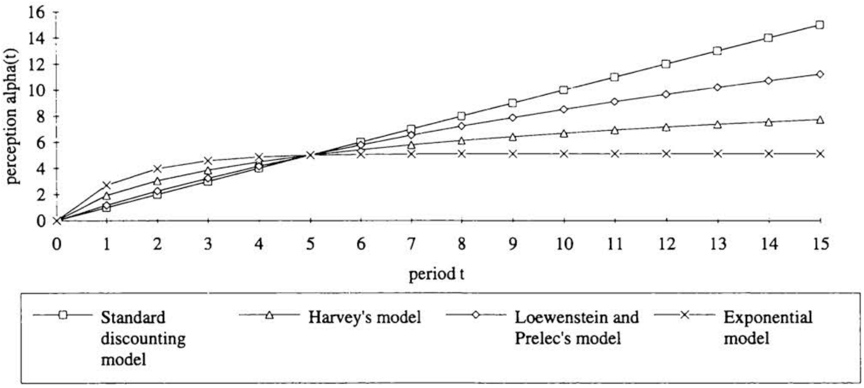

Figure 1: Time perception functions in various models of intertemporal choice

tionarity is presented as a condition of rationality, it is mostly interpreted as a dynamic consistency condition (see, e.g., Chew and Epstein, 1990).

We thus need to present and analyse Strotz' argument in detail. We will show that, under certain conditions, Strotz' dynamic consistency is indeed equivalent to the stationarity axiom. These 'certain conditions', however, have mostly been neglected in the time preference literature. We feel that there exist many decision contexts where these conditions are far from being obviously satisfied. We will argue that they are not satisfied, e.g., when decisions are binding for the future. In these instances, dynamic consistency arguments are inappropriate to apply, and thus, stationarity is not a requirement of rationality.

Consider Figure 2. An individual prefers a_{t+l} = DM 1100 to b_{s+l} = DM 1000. To emphasise the period 0 the preference statement refers to, we index the symbol $\succeq$ and write $a_{t+l} \succeq_0 b_{s+l}$ and we wish to show that necessarily, $a_t \succeq_0 b_s$. Given his preference, the individual chooses to receive DM 1100 in period $t + l$. Then time passes and period l comes. The individual reconsiders his original choice (and we assume that he is still free to change his earlier decision). From the perspective of period l, period $t + l$ is now t periods away and period $s + l$ is now s periods away. He thus reframes the alternatives a_{t+l} and b_{s+l} he has chosen between in period 0 into a_t and b_s to choose between in period l. Dynamic consistency demands that he confirm his earlier decision, that is, that, in period l, $a_t \succeq_l b_s$. Now suppose that the decision maker's time preference has not changed since period 0. Then it follows that, *whenever* a is t periods away from the period the decision maker is in,

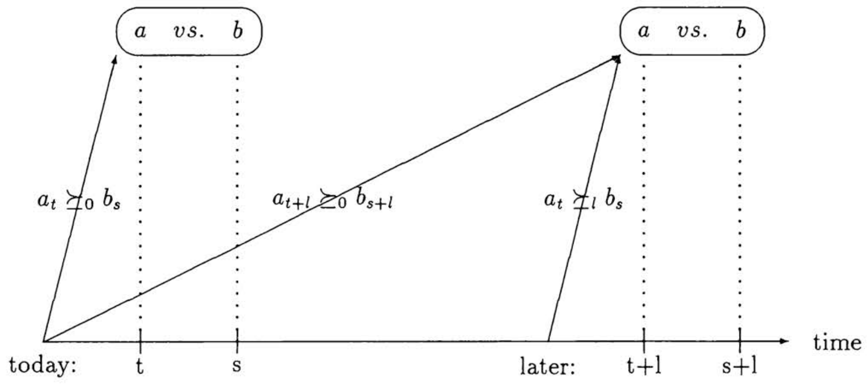

Figure 2: An illustration of dynamic consistency

while b is s periods away, he prefers a. In particular, $a_t \succeq_0 b_s$. We have thus shown how to use a dynamic consistency argument to derive the stationarity condition. We now formalise these ideas.

Axiom 6 (Dynamic consistency)

For all $t, s, l \in N_0$ *and all* $a, b \in X$, $a_{t+l} \succeq_0 b_{s+l} \;\Leftrightarrow\; a_t \succeq_l b_s$.

Dynamic consistency is widely accepted as a rationality requirement. A dynamically inconsistent individual could keep on changing his decisions, not always doing what he decided to in the past.

Besides dynamic consistency, in the above example, we required that the individual's time preference did not change between periods 0 and l. Formally,

Axiom 7 (Constant time preferences)

For all $t, s, l \in N_0$ *and all* $a, b \in X$, $a_t \succeq_0 b_s \;\Leftrightarrow\; a_t \succeq_l b_s$.

We are now in a position to state Strotz' result:

Theorem 10 (Strotz, 1956)

Suppose that in each period, the decision maker's preferences are described by the intertemporal value function (3) and suppose Axiom 7 holds. Then, V takes on the form (4) if and only if Axiom 6 holds.

Having formally defined the assumptions of Strotz' theorem, it is obvious to see why Strotz arrives at the same result as Koopmans (1960).

Since the right hand sides of the equivalence statements in Axioms 6 and 7 coincide, we can plug them together to find

Theorem 11

Suppose that Axiom 7 holds. Then the decision maker's preference in period 0 is stationary if and only if his preferences are dynamically consistent.

Under Axiom 7, Strotz' (1956) and Koopmans' (1960) results are thus reformulations of one another, leading to the standard discounting model and to linear time perception.

Theorem 11 gives stationarity an interpretation as a rationality requirement. Reconsidering the example we have used to illustrate how dynamic inconsistencies can be derived from nonstationary behavior, we see, however, that this interpretation has made some tacid assumptions about the decision context that need not necessarily be satisfied.

First, Axiom 7 does not necessarily hold. If it does not, dynamic consistency and stationarity are no longer equivalent. Thus, Axiom 7 is crucial for the argument that stationarity is a requirement of dynamic consistency. Indeed, time preferences will change with age and circumstances. Which personal factors influence individual time preferences – and how – was subject of early psychological contributions to the time preference literature that date back to the nineteenth century, see, e.g., von Böhm-Bawerk's (1889) influential work. Even without Axiom 7, the question of which present decisions will be obeyed in the future still is an important issue, except that it does not have a simple answer.

Despite this fact, a case for Axiom 7 can be made. Dynamic consistency demands that present and future decisions coincide. In order to verify dynamic consistency, one therefore needs to assume that future preferences are already known today. Of course, one may argue that this assumption is unsound. Then, however, one denies that dynamic consistency is a meaningful concept at all. In light of the influence Strotz' work had on the development of economic thought, few economists would share this view. On the other hand, if one accepts that some assumption about future preferences needs to be made, then it is straightforward to make the canonical assumption that preferences do not change over time, in which case one arrives at Axiom 7.

More important is the following second observation: not all present decisions will be reevaluated later and, even if they will, the decision maker will not necessarily have the option to change his earlier decision. Managerial decisions very often consist of seizing or neglecting an opportunity that offers itself at that particular moment but that may be gone a few periods later. Such decisions include mergers and acquisitions, buying real-estate to have the future option to enlarge production facilities or hiring a particular applicant for some position in the firm. Other decisions may be binding for the future, the closing down of production facilities usually is irrevocable, as is the decision to go public. In politics, it would not be easy to change the decision to form a currency union within the EC once a single currency had been introduced, nor could the decision of the West German government in 1989 to agree to a currency union with East Germany be changed today. The decision on countermeasures to be taken against the nuclear release from the Chernobyl accident had to assume that whatever their consequences, they would be irrevocable: once exposed to radiation, each individual would suffer a given increased risk of cancer. On the level of personal decision making, the decision to quit a job or sell a house usually is irrevocable. The decision to accept a job offer or buy a house may be changed later on, though possibly at considerable cost and distress.

On the other hand, consumption decisions have to be made every day, and thus, plans for future consumption can be – have to be – regularly confirmed or changed. Capital markets transactions have to be reevaluated on a regular, daily, or even real-time basis. For these decisions, the question of dynamic consistency is an important issue.

For decisions that are binding for the future, however, dynamic consistency is a meaningless concept. Then so is stationarity.

What other justification – except dynamic consistency arguments – would there be to assume stationarity as the only basis of a prescriptive analysis? If a decision maker happened to find his preferences *correctly described* by the stationarity axiom, he would have reason to assume it, not in order to obey what is considered rational, but in order to simplify his task of deriving an intertemporal value function. If dynamic consistency arguments are inadequate for the lack of future reevaluations, we can think of no other justification of stationarity than descriptive validity. Then the question is whether there are other axioms that are descriptively more valid and further simplify (2).

If there were, the decision maker could use them instead of stationarity. It all depends on his personal time preference and perception. In the Appendix, we present a procedure for eliciting the personal time perception function α and the parameter i.

In the following subsection, we will present alternatives to the stationarity axiom. Unlike stationarity, we do not judge the axioms on their rational appeal for they are not meant to be conditions of rationality. Rather, we will in subsection 3.4 judge and compare their descriptive validity.

3.3 Concave time perception

Despite the discussion about dynamic consistency, one may still be inclined to consider linear perception of the future as the only acceptable form of time perception. Of course, a year is a year, and shifting a one year waiting interval into the future does not take away a single one of the 12 months of that year. However, one of the central ideas in economics is that the value of an extra unit of a good depends on how much of that good one possesses. This concept of declining marginal value is operationalized by concave value functions. There is no a-priori reason not to transfer this basic idea to time preferences, allowing the possibility of non-linear, that is, concave time perception functions. We will now present some models of intertemporal choice which correspond to concave αs.

3.3.1 Harvey's model

Harvey (1986) suggested the following axiom as an alternative to stationarity:

Axiom 8 (Harvey condition)

For all $t, s, k \in N_0$, $k \neq 0$, *and all* $a, b \in X$, $a_t \succeq b_s \Leftrightarrow a_{k(t+1)-1} \succeq b_{k(s+1)-1}$.

When interpreting Axiom 8, and all formulas following from it, we ignore all plus and minus 1s and suggest the reader to do alike.[10]

As in the formulation of stationarity, both preference statements in this axiom are made in period 0. Assuming Axiom 8, all trade-offs only depend on the ratio $t + 1/s + 1$ of the two periods t and s in question since this quotient is left invariant by a stretch, i.e. a multiplication with k. It is in this sense that Harvey's model is a natural analogue to standard discounting: both models imply period trade-offs between t and s to be simple functions of t and s. In standard discounting, as we have seen, trade-offs depend on $t - s$, in Harvey's model, on $t + 1/s + 1$. Just like stationarity, Harvey's axiom is an ordinal concept.

[10] The plus and minus 1s stem from a technicality: Harvey denotes the first period he considers (the present) $t = 1$, while we call it $t = 0$. Adding or subtracting one then just means translating into or from Harvey's notation. We could have adopted Harvey's notation in order to simplify exposition of his model, at the cost of complicating exposition elsewhere. For instance, in (3) and (4), we would have had to add minus 1s in the exponents.

Theorem 12 (Harvey, 1988)

Suppose that (3) is an intertemporal value function. Then Axiom 8 holds if and only if there exists an $h \in R$, such that

$$V(a_0, \ldots, a_{T(a)}) = \sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+t)^h}. \tag{5}$$

Here, the period decision weights decline as a hyperbolic function in t. This class has therefore been known as hyperbolic discounting (Ainslie, 1975, Ainslie and Haslam, 1992) and has given its name to all models of intertemporal choice with discount factors decreasing over time. The intuitive interpretation of the parameter will become clear once we have transformed (5) to make it compatible with our model (3).

The time perception function Harvey's model corresponds to is

$$\alpha(t) = h \frac{\ln(1+t)}{\ln(1+i)}$$

since

$$\begin{aligned}
\sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+i)^{\alpha(t)}} &= \sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+i)^{\frac{h\ln(1+t)}{\ln(1+i)}}} \\
&= \sum_{t=0}^{T(a)} \frac{v(a_t)}{\left((1+i)^{\log_{(1+i)}(1+t)}\right)^h} \\
&= \sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+t)^h}.
\end{aligned}$$

Impatience goes along with a positive i. Then $\ln(1+i)$ is positive and since α is positive, patience is equivalent to a positive h. Conversely, patience corresponds to a negative i, thus a negative $\ln(1+i)$ and a negative h. Timing indifference corresponds to $h = 0$.

We can now give h an intuitive meaning: h is a linear factor of the time perception function α. The higher h, the longer will one time unit be perceived to last. This means that h directly measures the speed of our time perception. Consider the extreme values h may take: if $h = 0$, periods are perceived to pass infinitely fast. If this is so, waiting is not aversive at all. Then we would expect the decision maker to be timing indifferent, i.e., we would expect all period weights to be identical. Indeed, then $1/(1+t)^h = 1/(1+t)^0 = 1$ for all t. For $h \to \infty$, time is perceived not to pass at all. Then we cannot derive value from future consequences and the decision weight of all periods $t > 0$ should be zero.

Correspondingly, in the Harvey model, $1/(1+t)^h$ approaches 0 for all periods $t > 0$. Conversely, if h is negative the decision maker is patient; and as $h \to -\infty$, decision weights $1/(1+t)^h$ of future periods $t > 0$ tend to infinity, as intuition suggests.

By concavity of Harvey's α we know that a given time interval is perceived to be shorter if it is pushed into the future. We ask by how much a given time interval must be prolonged in order to compensate for non-linear time perception. For all $k \neq 0$ we have

$$\begin{aligned}\alpha(k(t+1)-1) - \alpha(k(s+1)-1) &= \frac{h}{\ln(1+i)}\big(\ln(k(t+1)) - \ln(k(s+1))\big)\\ &= \frac{h}{\ln(1+i)}\big(\ln(t+1) - \ln(s+1)\big)\\ &= \alpha(t) - \alpha(s)\end{aligned}$$

What the above formula shows is that the interval $[t, s]$ is perceived to be equal to $[k(t+1)-1, k(s+1)-1]$). For instance, the trade-off between periods $s = 0$ and $t = 1$ is the same as between periods $k(s+1)-1 = 10$ and $k(t+1)-1 = 21$ (stretch by a factor $k = 11$). The Harvey model therefore assumes our concave time perception to follow a very simple rule. If a given time interval is pushed into the future, a stretch proportional to how far it lies in the future will leave the perception of that interval constant. Figure 1 displays a Harvey α normalised such that $\alpha(0) = 0$ and $\alpha(5)$ is identical to those of the other models presented here.

3.3.2 Loewenstein and Prelec's model

Loewenstein and Prelec (1992, p. 125) suggest to assume that 'the delay that compensates for the larger outcome is a linear function of the time to the smaller, earlier outcome'. Formally

Axiom 9 (Loewenstein and Prelec condition)

For all $t, s, l \in N_0$, there exists a $k_{t,s} \in R$ such that for all $a, b \in X$, whenever[11] $l \cdot k_{t,s} \in N_0$, $a_t \succeq b_s \Leftrightarrow a_{t+l} \succeq b_{s+l \cdot k_{t,s}}$.

Again, this is an ordinal concept like stationarity and Harvey's axiom. Again, both preference statements refer to today. Here, it is only

[11] In order to derive their result, Loewenstein and Prelec require a continuous time axis. We have restricted attention to discrete time periods in order to simplify interpretation of the theory. For instance, with a continuous time axis, V would have to be written as an integral rather than as a finite sum. Due to the restriction to discrete time periods, we need to restrict attention to those cases where $l \cdot k_{t,s} \in N_0$.

required that the above preference statement holds for some $k_{t,s}$, which will depend on t and s. For different choices of $k_{t,s}$ as a function of t and s, Loewenstein and Prelec's axiom can either correspond to stationarity or to the Harvey condition. However, it is not so important to analyse which functional form of k implies which other axiom[12], but to understand what makes this formulation more flexible than other axioms.

Assuming the Loewenstein and Prelec condition, we arrive at a time perception determined by two parameters g and h. It is through this one additional parameter that Loewenstein and Prelec's model has an additional degree of freedom.

Theorem 13 (Loewenstein and Prelec, 1992)

Suppose that (3) is an intertemporal value function. Then, if for some $h, g \in R$, $g > 0$,

$$V(a_0, \ldots, a_{T(a)}) = \sum_{t=0}^{T(a)} \frac{v(a_t)}{(1 + gt)^{\frac{h}{g}}}, \tag{6}$$

or if V takes on the form (4), Axiom 9 holds. Conversely, if Axiom 9 holds, and if one assumes a continuous time axis, V either takes on the form (6) or (4).

For $g = 1$, we get Harvey's more specific model. The parameter h has the same meaning as in the Harvey model. The meaning of the parameter g will again become clearer through the corresponding time perception function. This is

$$\alpha(t) = \frac{h \ln(1 + gt)}{g \ln(1 + i)},$$

which can be checked by a calculation analogous to the one shown for the Harvey α.

In order to interpret g, consider the limiting case as g tends to zero. We have

$$\begin{aligned}
\lim_{g \to 0} \alpha(t) &= \lim_{g \to 0} \frac{h \ln(1 + gt)}{g \ln(1 + i)} \\
&= \frac{h}{\ln(1 + i)} \lim_{g \to 0} \frac{\frac{t}{1 + gt}}{1} \\
&= \frac{h}{\ln(1 + i)} t
\end{aligned}$$

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12 If $k_{t,s} = 1$ for all t and s, the axiom corresponds to stationarity. If $k_{t,s} = s + 1/t + 1$ for all t and s, it corresponds to Harvey's axiom.

As g approaches 0, Loewenstein and Prelec's time perception approaches linearity in t, thus the standard discounting model. Therefore, g can be interpreted as describing the departure of (6) from standard discounting. If $g = 0$ was admissible, both standard discounting ($g = 0$) and the Harvey model ($g = 1$) would be special cases of equation (6). Since division by $g = 0$ is not admissible, in Theorem 13, the corresponding limiting case, the standard discounting model (4), needs to be stated separately. We see that the Loewenstein and Prelec model covers a continuum of time perceptions between standard discounting and the Harvey model. Figure 1 displays a Loewenstein and Prelec α, intermediate ($g = 0.1$) between Harvey's and the linear α.

3.3.3 An exponential model

Within the general model of intertemporal choice the function α determines the decision maker's attitude towards time. This time perception function was only restricted to be strictly increasing. In principle, a large variety of time perception functions seems possible. Either a decision maker's time preference is assessed through simple questions (for an idea see the Appendix) and a piece-wise linear function is used or some other functional form is considered.

As an example, one could consider a functional form which has proved fruitful in the theory of decision under risk. We will ask what time preference model it leads to if one interprets it as a time perception function. We are aware that the vague analogue between time and risk preferences is only modest justification for using prominent utility functions as time perception functions, but we feel that it raises interesting research questions.

The exponential function $u(x) = k - le^{-mx}$, where $k, l, m \in R$, $m, l > 0$ are constants, has widely been applied since it implies constant absolute risk aversion. If interpreted as a time perception function, this suggests a model of intertemporal choice that has not been proposed in the literature so far:

$$V(a_0, \ldots, a_{T(a)}) = \sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+i)^{k - le^{-mt}}}.$$

Again, Figure 1 displays the corresponding time perception. It may be worthwhile to ask whether the concept of constant absolute risk aversion can, maybe via this model, be transferred to the theory of time preference.

3.4 Comparison of models

We will now fit the theoretical models to empirical data: we want to see what time perception function α can be expected to be a realistic description of behavior. This is important not only for descriptive, but also for prescriptive purposes. If, for the lack of future revisions, dynamic consistency and stationarity fail to have prescriptive appeal, the decision maker can – needs to – use his actual personal time perception. We restrict ourselves to standard discounting and to Harvey's model. Since the Loewenstein and Prelec model as well as the exponential one have one additional degree of freedom, it is not obvious how to compare them with either of the two other models.

We will use the two empirical studies[13] by Benzion et al. (1989) and Shelley (1993). We chose these studies for three reasons: the studies are based on a large enough sample of subjects (Benzion et al. 204, Shelley 74), they study time preference only for the case of certainty, and they only consider monetary outcomes in order not to have results for different outcome domains get entangled and obscure interpretation.

Figures 2 and 3 present the time perception functions $\alpha(t)$ and the resulting period weight functions w_t of both the theoretical models and the empirical studies[14].

Figure 2 shows that all models have increasing time perception functions α, thus assume impatient decision makers. The Harvey model as well as the Benzion et al. (1989) and Shelley (1993) data correspond to a concave α-curve below the standard discounted value α-curve.

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13 There are other experimental studies that have investigated individual time preference, the first we are aware of being Thaler's (1981). His subject pool was rather small ('about twenty usable responses', Thaler, 1981, p. 203). He found that discount rates decline with the size of the amount to be discounted, that they are lower for losses than for gains and that they decline as waiting time increases. Loewenstein (1988) established that individual rates differ for speeding up and delaying future receipt of different consumption goods. Stevenson (1992) found discount rates for risky prospects to be lower than for certain ones. Ahlbrecht and Weber (1995) showed that Stevenson's results as well as the result that discount rates decline as waiting time increases hold for matching, but not for choice tasks.

14 From Table 2, p. 278, of the Benzion et al. (1989) and Figure 4, p. 812, of the Shelley (1993) study, we have calculated $\alpha(t)$ and w_t values corresponding to mean discount rates for different time periods (6, 12, 24 and 48 months in both studies) over all scenarios, amounts and subjects. We have used linear interpolation for time periods intermediate between those used in these studies. Value functions have been assumed to be linear, as in both studies. We have taken the smallest time interval used there, half a year, to denote one period and normalized α such that $\alpha(0) = 0$ and $\alpha(1) = 1$. Since the w_ts are determined not only by α but also by $1 + i$, we have applied the average $1 + i$ resulting from the Benzion et al. (1989) and Shelley (1993) study to both theoretical models.

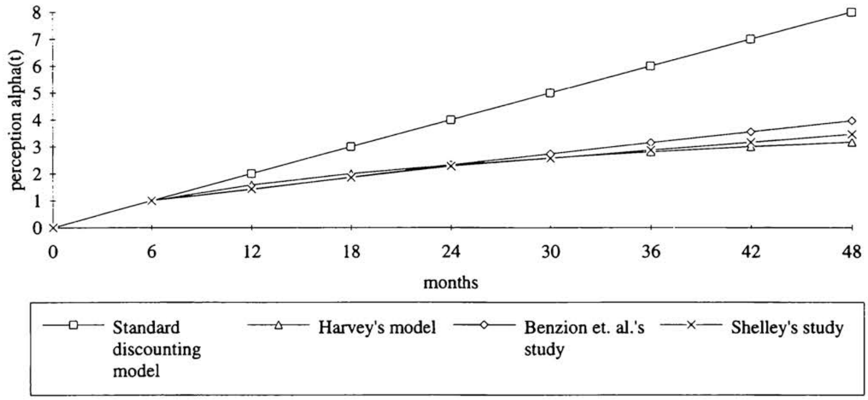

Figure 3: Theoretical and empirical time perception functions

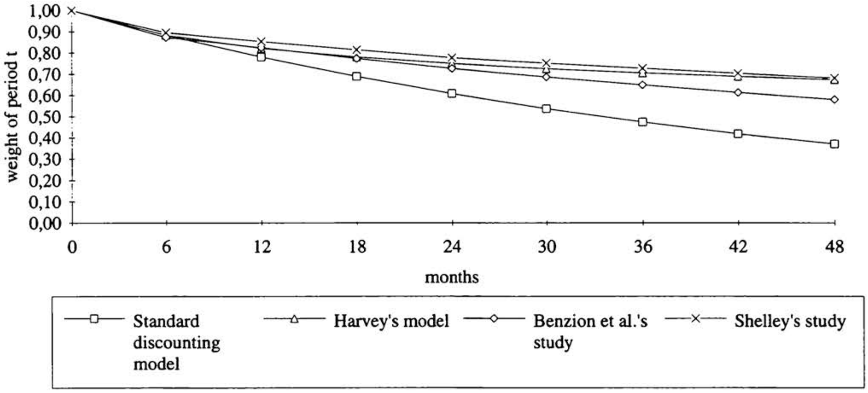

Figure 4: Theoretical and empirical period weight functions

Note that both Figure 2 and 3 indicate that the Benzion et al. (1989) and Shelley (1993) data are more consistent with hyperbolic than with standard discounting. Their plots are very similar and are obviously closer to the hyperbolic models. This supports the hypothesis that time perception is concave rather than linear.

4. Summary and Discussion

We have axiomatized a general intertemporal value function. This value function shows that many different forms of discounting are possible. Each of these forms can be linked to a corresponding time perception function that describes the perception of time intervals as a function of how far they lie in the future. Standard discounting and hyperbolic discounting are just examples of the general model.

We have shown that the dynamic inconsistency arguments that have been brought forward against hyperbolic value maximizers – against any nonstandard discounting model – are not always as convincing as the literature may suggest. Whenever irrevocable decisions have to be made, dynamic consistency arguments are meaningless. Thus it would be inadequate to resort to dynamic consistency arguments to derive an intertemporal value function for such types of decisions.

Savings-consumption decisions as well as financial market transactions are examples of decision contexts where dynamic consistency is an important issue and where, therefore, standard discounting is the only rational model of decision making. Many managerial, political or personal decisions, however, are binding for the future, and the irrevocability of their consequences refutes any attempt to deduce the standard discounted value model as the only rational approach for these decisions.

Rather, the individual decision maker has to derive period decision weights from his personal time preference (perception). A considerable simplification of this task is to find a descriptively valid specification of the decision weights where all w_ts depend on only a few parameters, possibly just one. If stationarity happens to correctly describe the decision maker's preferences, then the standard discounted value model should be used regardless of dynamic consistency arguments. Experimental studies, however, suggest that alternative axioms of intertemporal choice, leading to hyperbolic time preference models, describe actual time preference better than does stationarity while keeping the analysis as tractable as in the standard model. We therefore suggest that hyperbolic discounting models offer themselves as a valid approach for supporting rational decision making whenever decisions are binding for the future.

All our results are restricted to certainty. The interesting question which of our results would still hold, if, more realistically, future consequences were taken to be uncertain, remains open. In future research it will be interesting to combine ideas of choice under risk and of intertemporal choice to hopefully derive a richer set of models to help to make optimal decisions in the case of risky and intertemporal settings.

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A. Elicitation of parameters

First of all, the individual needs to assess his value function v. Procedures for eliciting v have been extensively discussed in the literature and will not be discussed here, see von Winterfeld and Edwards (1986), Farquhar and Keller (1989) or Eisenführ and Weber (1994) for an overview.

We will assume that the perception of the first waiting period is normalized to 1 by letting $\alpha(0) = 0$ and $\alpha(1) = 1$. This restriction is admissible since α is determined only up to linear transformations. The parameter i can be elicited via indifference statements between consequences in periods 0 and 1. Suppose $a_0 \sim b_1$. Then we can calculate i from

$$\frac{v(a)}{(1+i)^0} = \frac{v(b)}{(1+i)^1}, \text{ i.e. } i = \frac{v(b)}{v(a)} - 1.$$

For the elicitation of α further indifference statements between future periods may be necessary, depending on how many parameters determine α. For the standard discounting model, no further parameter needs to be assessed. The same is true for Harvey's model, since through $\alpha(1) = 1$ we can calculate the only parameter h from

$$\frac{v(b)}{(1+i)^1} = \frac{v(b)}{(1+1)^h}, \text{ i.e. } h = \frac{\ln(1+i)}{\ln(2)}.$$

For the exponential and the Loewenstein and Prelec models, the parameters and thus α can be determined by an indifference statement $a_0 \sim b_t$ for an arbitrary t, say $t = 2$, through

$$\frac{v(a)}{(1+0)^0} = \frac{v(b)}{(1+t)^{\alpha(t)}}, \text{ i.e. } \alpha(t) = \frac{\ln v(b) - \ln v(a)}{\ln(1+t)}.$$

Eliciting parameters from indifference statements between periods 0, 1 and 2 will be sensitive to small errors in the individual's answers. Such errors will accumulate and may lead to severe misspecifications of future periods' weights. We therefore suggest a consistency check as proposed by Harvey (1986). The individual should first specify the consequence a that has value $v(b)/2$, where b is the best possible period consequence. He should then ask himself to which period h the optimal b must be delayed to make him indifferent between $a_0 \sim b_h$. This period h can be called the temporal midvalue. We have

$$\frac{v(a)}{(1+0)^0} = \frac{v(b)}{(1+t)^{\alpha(h)}}, \text{ i.e. } \alpha(h) = \frac{\ln 2}{\ln(1+i)}.$$

The consistency check should then verify whether the parameters elicited are consistent with $\alpha(h)$. If yes, they can safely be used. If not, they have either to be reelicited, or maybe even, a different parametrization of α should rather be used.

B. Proofs

Theorem 1 If V is a continuous ordinal intertemporal value function, completeness, transitivity and continuity of the ordinal preferences can be checked directly. Conversely, if all these conditions hold, by Debreu (1954, Theorem I, p. 162), a continuous ordinal intertemporal value function exists. For the uniqueness part, see Debreu (1960, Theorem I, p. 18).

Theorem 2 If V takes on the form (1), Axiom 1 can be checked directly. Conversely, if Axiom 1 holds, V takes on the form (1) by (Debreu, 1960, Theorem 3, p. 21) or (Wakker, 1989, Theorem III. 4.1, p. 49), who also prove the uniqueness part.

Theorem 3 If V takes on the form (2), Axiom 2 can be checked directly. Conversely, if Axiom 2 holds, V takes on the form (2) by (Dyckhoff, 1988, Anhang 2, p. 1007).

Theorem 4 If V is a continuous cardinal intertemporal value function, completeness, transitivity and continuity both of the ordinal and cardinal preferences as well as equations (i), (ii) and (iii) can be checked directly. Conversely, suppose that all these conditions hold. We wish to apply Theorem 2 of Debreu (1960, p. 19). First, we need to verify that Assumptions 2.1, 2.2 and 2.3 of Debreu's theorem hold.

As for Assumption 2.1, define $A := \{b \to a \in \mathcal{X}^2 \mid a \succeq b\}$. By Theorem 1, there exists a continuous function $F : A \to R$ which represents the cardinal preferences on A. Extend F to $\mathcal{X}^2$ by letting $F(ab) = -F(ba)$ whenever $a \prec b$ (and thus $a \to b \in A$). By (iii), for all $a, c \in \mathcal{X}$, $a \to a \sim c \to c$ such that we may normalize F to have $F(a \to a) = 0$ for all $a \in \mathcal{X}$. Thus the extension is well-defined and continuous. By (i), the extended F represents the cardinal preferences on the whole of $\mathcal{X}^2$. Define $P := F + 0.5$. Then P satisfies Debreu's Assumption 2.1. Assumption 2.2 is equivalent to our assumption (ii). Assumption 2.3 holds since $\mathcal{X} \times \{a\}$ is connected for all a, and then so is $P(\mathcal{X} \times \{a\})$ since P (as defined above) is continuous.

By (Debreu, 1960, Theorem 2, p. 19), there is a function $U : \mathcal{X}^2 \to R$ the value differences of which represent the cardinal preferences. By Theorem 1, there is an ordinal intertemporal value function V. It remains to show that U represents the ordinal preferences, too. By (c), for all $a, b, c \in \mathcal{X}$, $V(a) \geq V(b) \Leftrightarrow U(a) - U(b) \geq U(c) - U(c) = 0 \Leftrightarrow U(a) \geq U(b)$.

Theorem 5 See Dyer and Sarin (1979, Theorem 1, p. 813).

Theorem 6 If V takes on the form (2), Axiom 4 can be checked directly. Conversely, suppose that Axiom 4 holds. Write $v_0 = v$. By Axiom 4, all v_t describe the same cardinal preferences for period consequences. Then by Theorem 4, for each $t \in N_0$, there exist $k_t, l_t \in R$, $l_t > 0$ such that for all $x \in X$, $v_t(x) = k_t + l_t\, v(x)$. Then, (1) reduces to

$$\begin{aligned} V(a_0, \ldots, a_{T(a)}) &= \sum_{t=0}^{T(a)} w_t v_t(a_t) = \sum_{t=0}^{T(a)} w_t (k_t + l_t v(a_t)) \\ &= \sum_{t=0}^{T(a)} w_t k_t + \sum_{t=0}^{T(a)} w_t l_t\, v(a_t) \end{aligned}$$

Write $w'_t = w_t l_t$ for all t. Then, the value function $V'(a_0, \ldots, a_t) = \sum_{t=0}^{T(a)} w'_t v(a_t)$ is a positive linear transformation of V and thus equivalent to V.

Theorem 7 We have

$$\sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+j)^{\alpha'(t)}} = \sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+j)^{k+l\alpha(t)}} = \frac{1}{(1+j)^k} \sum_{t=0}^{T(a)} \frac{v(a_t)}{(1+i)^{\alpha(t)}}$$

Then by Theorem 4, V and V' describe the same (ordinal or cardinal) preferences. Conversely, suppose V' describes the same (ordinal or cardinal) preferences as does V. Then by Theorem 2, there exists an $r \in R$, $r > 0$, and for all $t \in N_0$ there exist $k_t \in R$ such that for all $x \in X$,

$$\frac{v(x)}{(1+i)^{\alpha(t)}} = r \cdot \frac{v(x)}{(1+j)^{\alpha'(t)}} + k_t.$$

Letting $x = n$, it follows that $k_t = 0$. Letting $x \neq n$, it follows that

$$\alpha'(t) = \alpha(t) \cdot \log_{1+j}(1+i) + \log_{1+j} r.$$

Define $l := \log_{1+j}(1+i)$ and $k := \log_{1+j} r$. The theorem follows.

Theorem 8 is immediate.

The proofs of Theorems 9, 12 and 13 have been given in the original papers. Since notation differs throughout the literature, we give separate proofs of these theorems here. To do so, we need the following

Lemma Suppose that (3) is an (ordinal or cardinal) intertemporal value function. Let $q, r, s, t \in N_0$. Then the following two statements are equivalent:

(i) For all $a, b \in X$, $a_t \succeq b_s \Leftrightarrow a_r \succeq b_q$

(ii) $\alpha(s) - \alpha(t) = \alpha(q) - \alpha(r)$.

Proof Suppose that (i) holds. Choose $a, b \in X$, $b \neq n$, such that $a_t \sim b_s$. Then $\frac{v(a)}{(1+i)^{\alpha(t)}} = \frac{v(b)}{(1+i)^{\alpha(s)}}$ and by (i), $\frac{v(a)}{(1+i)^{\alpha(r)}} = \frac{v(b)}{(1+i)^{\alpha(q)}}$. From these equations, (ii) follows. The converse implication follows directly.

Theorem 9 If V takes on form (4), Axiom 5 is readily verified. Conversely, suppose that Axiom 5 holds, then for all $a, b \in X$, $a_0 \succeq b_1 \Leftrightarrow a_1 \succeq b_2 \Leftrightarrow \ldots a_t \succeq b_{t+1}$. Thus by the above Lemma and by an induction, for all $t \in N_0$, $\alpha(t) = t \cdot \alpha(1)$. By Theorem 7 we may assume that $\alpha(1) = 1$.

Theorem 10 follows from Theorem 11 and Theorem 9.

Theorem 11 is immediate.

Theorem 12 In his original paper, Harvey (1986, Theorem 5, p. 1129) proved his theorem only under the assumption of impatience, thus for $h > 0$. Our formulation and proof of his theorem is thus more general. If V takes on form (5), Axiom 8 is readily verified. Conversely, suppose that Axiom 8 holds. Then for all $a, b \in X$

and $t, n \in N_0$, $a_0 \succeq b_{t-1} \Leftrightarrow a_{t-1} \succeq b_{t^2-1} \Leftrightarrow \ldots a_{t^n-1} \succeq b_{t^{n+1}-1}$. Thus by the above Lemma and by an induction, $n \cdot \alpha(t-1) = \alpha(t^n - 1)$ for all $n \in N_0$.

Choose $h := 1/\log_{1+i} 2$. By Theorem 7, we may assume that $\alpha(1) = 1$. It follows that for all $n \in N_0$, $\alpha(2^n - 1) = n = h \cdot \log_{1+i} 2^n$.

It remains to show that for all $t \in N_0$, $\alpha(t-1) = h \cdot \log_{1+i} t$. Let $\epsilon > 0$. We show that $|\alpha(t-1) - h \cdot \log_{1+i} t| < \epsilon$. Choose an $n \in N_0$ such that $1/n < \epsilon$. Choose $m \in N_0$ such that $2^m - 1 < t^n - 1 < 2^{m+1} - 1$. Since α and log are strictly increasing functions, $\alpha(t^n - 1), h \log_{1+i} t^n \in [\alpha(2^m - 1), \alpha(2^{m+1} - 1)] = [h \log_{1+i} 2^m, h \log_{1+i} 2^{m+1}]$. Then $|\alpha(t-1) - h \cdot \log_{1+i} t| = 1/n \cdot |\alpha(t^n - 1) - h \cdot \log_{1+i} t^n| \leq (h/n) \log_{1+i} 2 = 1/n < \epsilon$.

Theorem 13 Suppose that V takes on the form (6). For all $t, s \in N_0$, let $k_{t,s} = \dfrac{1 + gs}{1 + gt}$. A straightforward calculation verifies Axiom 9. Suppose that V takes on the form (4). For all $t, s \in N_0$, let $k_{t,s} = 1$. Then, too, Axiom 9 holds. For the converse implication, which we do not derive here for the lack of a continuous time axis, see Loewenstein and Prelec (1992, p. 126).

Zusammenfassung

Dieser Beitrag zeigt auf, daß Nichtstandarddiskontmodelle als präskriptive Zeitpräferenztheorien dienen können, wenn eine für die Zukunft bindende Entscheidung zu treffen ist. Zunächst entwickeln wir eine einheitliche axiomatische Basis für Standard- und Nichtstandarddiskontmodelle. Die Entscheidungsgewichte ergeben sich als $1/(1+i)^{\alpha(t)}$, wobei $\alpha(t)$ die Zeitwahrnehmung des Entscheiders abbildet. Das Standarddiskontmodell entspricht einem linearen α, Nichtstandarddiskontmodelle dagegen entsprechen nichtlinearen αs. Anschließend begegnen wir einem Inkonsistenzargument, das gegen Nichtstandarddiskontmodelle vorgebracht wurde. Wir zeigen, daß die Überzeugungskraft dieses Arguments, und damit die Anwendbarkeit von Nichtstandarddiskontmodellen auf präskriptive Theorien, von dem jeweiligen Entscheidungskontext abhängt.

Abstract

This paper argues that hyperbolic discounting models can be used in prescriptive theory of intertemporal choice whenever decisions are binding for the future. First, we derive an axiomatic basis that unifies standard and hyperbolic discounting models. The decision weights are written as $1/(1+i)^{\alpha(t)}$ where $\alpha(t)$ is a time perception function. The standard discounting model corresponds to a linear α whereas hyperbolic models refer to nonlinear αs. Second, we make a qualification to an inconsistency argument brought forward against hyperbolic discounting models. We show that the strength of this argument, and thus, the applicability of hyperbolic models to prescriptive theory, depends on the decision context.

JEL-Klassifikation: D9