Zeitschrift für Wirtschafts- u. Sozialwissenschaften (ZWS) 107 (1987), S. 581 - 594
Duncker & Humblot, Berlin 41

Steady State Growth Paths in a Continuously Overlapping Generations Model

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This paper introduces an overlapping generations model into the neoclassical growth theory. All individuals are identical with respect to preferences and expectation of life, but differ in their date of birth. They decide on the optimal paths of consumption and leisure within their time horizon. By aggregating the economic activities of all living generations we are able to analyze the macro effects of all micro decisions. The implications of Harrod-neutral technical progress on the existence and economic properties of a market equilibrium complete the analysis. Additionally, the golden rule of accumulation is examined.

Introduction

The usual topic of the neoclassic growth theory is to deduce the properties of steady state paths and to describe special kinds of policies, for instance by an optimal choice like the well-known golden rule of accumulation.[1] However, until now those macro-economic results are only supported by a micro-economic analysis of individual behavior in the absence of technical progress and without modelling labor as an endogenous variable.

Let us look at a growing economy in which the population consists of several generations, each living T years. The decision to consume or invest is due to the maximization of life-cycle utility and is based on the concept of a representative consumer for each generation. As long as we neglect technical progress all well-known results of the macro-economic growth theory can be obtained for arbitrary preference orderings; the only assumptions to be made is the identity of all generations' tastes. This result changes rapidly if we explicitly introduce Harrod-neutral technical progress which is labor augmenting. Then steady state growth paths require a special form of the utility function. Such a strong supposition leads to inevitable assumptions on the kind of the benefit functional.

Until now economists asserted it would be sufficient for generating a steady state growth path that there is either an additively separable or a

* This research was supported by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303. I would like to thank my collegues at the Bonn research seminar in public finance and an anonymous referee of this journal for helpful suggestions.

[1] *Phelps* (1961).

homothetic utility function if technical progress takes place.[2] The arguments are merely convincing if consumption is the only decision variable. However, if there are additional variables, expecially labor, more restrictive assumptions on the preference relation are necessary.

I. Individual Life-Cycle Decisions

Each individual born at θ lives T years till $\theta + T$. It maximizes a benefit functional[3]

$$B(\theta) = \int_{\theta}^{\theta + T} U(c(\theta, t)\,;\, 1 - l(\theta, t)) e^{-\delta t} dt \,, \tag{1}$$

which is the discounted sum of utility at each moment t; time preference is given by δ. The life time benefit depends on consumption $c(\theta, t)$ and leisure, defined as total amount of time available 1 minus labor $l(\theta, t)$. Generation θ's control problem therefore reads

$$\begin{aligned} &\max \quad B(\theta) \\ &s.t. \int_{\theta}^{\theta + T} s(\theta, t) e^{-rt} dt = 0 \\ &\text{or} \\ &V(\theta, \theta) = V(\theta, \theta + T) = 0 \\ &\dot{V}(\theta, t) = s(\theta, t) e^{-rt} = [w(t) l(\theta, t) - c(\theta, t)] e^{-rt} \,. \end{aligned} \tag{2}$$

According to the budget constraint the sum of discounted savings from labor income $s(\theta, t)$ is zero. An alternative control formulation uses wealth $V(\theta, t)$ as a state variable following the respective path, and there are no bequests. We assume a perfect capital market, therefore the borrowing and lending rates are identical to the market interest factor r.

The solutions of such a decision problem describe the optimal choice of consumption, labor and the respective savings within the whole time horizon. All macro-economic results are based on these individual decisions.

II. Overlapping Generations

Let us analyze those effects which occur on the level of aggregated economic measures. While a single consumer does not influence the

[2] *Gale* (1973), *Samuelson* (1975).

[3] $B(\theta)$ is an ordinal functional while $U(\theta)$ is cardinal, because the utility of each period is intertemporally comparable.

economic equilibrium in a distinct way, all living generations together are an important factor in describing the behavior of an economy.

Therefore, the analysis is based on the behavior of the different generations. Any generation consists of $P(\theta)$ identical individuals. The number of births is growing at a constant rate w_p. Therefore $P(\theta)$ can be written as

$$P(\theta) = P_o e^{w_p \theta} . \tag{3}$$

To discuss the growth model we have to aggregate the individual choice of all generations.[4] Hence we define the following macro-economic variables. The total savings from labor income are

$$S(t) = \int_{t-T}^{t} P(\theta) s(\theta, t) d\theta . \tag{4}$$

Overall consumption at time t is

$$C(t) = \int_{t-T}^{t} P(\theta) c(\theta, t) d\theta . \tag{5}$$

Finally, the total labor force of all generations living at time t is given by[5]

$$L(t) = \int_{t-T}^{t} P(\theta) l(\theta, t) d\theta . \tag{6}$$

If a steady state exists, the aggregated quantities (4), (5) and (6) can be written as

$$X(t) = X_o \; e^{(w_p + w_x)t} \qquad \text{for } x \in \{s, c, l\} , \tag{7}$$

where w_x is a micro-economic growth rate.

Lemma: The stationarity condition (7) will be fulfilled if the optimal individual choice of all generations can be reduced to the decision of a representative generation, i.e.

$$x(\theta, t) = e^{w_x \tau} x(\theta - \tau, t - \tau) . \tag{8}$$

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[4] The theory of overlapping generations goes back to Samuelson (1958); an extensive survey in this field is given by *Balasko / Shell* (1980, 1981a, b). A similar way of analyzing continuous time models can be found in *Elbers / Weddepohl* (1986).

[5] If there is a fixed age of retirement R the limits of integration change to $|_{t-R}^{t}$. Hence, the analysis of a public pension scheme can be integrated. For details see *Peters* (1987).

Proof:

$$X(t) = \int_{t-T}^{t} P(\theta) x(\theta, t) d\theta \tag{9}$$

or, by applying (8) and (3)

$$X(t) = P_o \int_{t-T}^{t} e^{(w_p + w_x)\theta} x(0, t - \theta) d\theta \ . \tag{10}$$

By changing the limits of integration, we get

$$X(t) = P_o \quad e^{(w_p + w_x)t} \int_{o}^{T} e^{-(w_p + w_x)\phi} x(0, \phi) d\phi \tag{11}$$

which completes the proof immediately. #

Condition (7) requires special properties of the generations' decision making. While all generations are identical in their preferences, they differ in their date of birth. Therefore, we assume that in a continuous time model the demand and supply functions should vary continuously from generation to generation.

As long as we are only interested in steady state growth paths, all economic parameters will grow at a constant rate. Thus, it is an unsurprising assumption that the individual's economic activities should grow at constant rates, too.

III. Neoclassical Growth Model

Net domestic product Y is produced by a linearly homogeneous, strictly quasi-concave production function in capital K and labor L. Additionally, technical progress Π will influence this NDP positively; it is assumed to be Harrod-neutral[6] or labor augmenting[7] which leads to

$$Y = f(\Pi \cdot L, K) = \Pi L \quad g\left(\frac{K}{\Pi L}\right) \ . \tag{12}$$

Hence, the usual equilibrium growth conditions can be derived:[8]

$$w_Y = w_K = w_\Pi + w_L \ , \tag{13a}$$

$$w_\Pi = w_w \ . \tag{13b}$$

[6] *Harrod* (1937).
[7] *Robinson* (1937/38), *Uzawa* (1960/61).
[8] *Krelle* (1985).

The macro-economic growth rates of labor-supply, w_L, and technical progress, w_Π, sum up to that of capital, w_K, and of NDP, w_Y. The prices will be influenced by this process in such a way that the real wage rate w grows at the same rate as the technical progress. The last result is based on the labor augmenting property of the technical progress. This means, for labor supply held constant the efficiency units of labor are growing. Hence, if each efficiency unit is payed a constant price, the wage rate has to follow the same development as the technical progress.

Additionally, the demand side of the NDP falls under two heads, the aggregated savings and consumption

(14) $$Y(t) = S(t) + C(t) ,$$

which leads to an equilibrium growth path with

(15) $$w_Y = w_S = w_C .$$

Taking together (13a), (15) and the basic lemma which connects macro with micro growth rates, we obtain the following properties

(16) $$w_K = w_p + w_s = w_p + w_c = w_w + w_p + w_l .$$

Therefore in the equilibrium the capital stock of the whole economy can be written as

(17) $$K(t) = K_o e^{(w_p + w_s)t} .$$

The above equation immediately shows the two main origins of capital growth. A higher rate of development is brought about (i) by a growing population, and (ii) by the individuals' amount of savings.

IV. No Technical Progress and Arbitrary Tastes

In the absence of technical progress ($w_\Pi = 0$) the wage rate will be constant over time. Therefore all generations face the same price system and their optimal plans do not differ from each other because (8) requires identical utility functionals of the generations. Hence, we derive the following properties of an equilibrium growth path without technical progress

(18a) $$w_Y = w_K = w_p ,$$

(18b) $$w_s = w_c = w_l = 0 .$$

All macro-economic growth rates are determined by the growth of the population, all micro-economic rates (18b) vanish because there are no dynamics in the price system. Hence, preferences may be arbitrary.

V. Technical Progress and Special Tastes

If we now introduce Harrod-neutral, labor augmenting technical progress, there does not only result a change in the growth rates, but also in the condition for the utility function such that the basic lemma, i.e. equation (8) is fulfilled. Starting with the equilibrium path characterized by (16) and (13b) it can be seen that

$$w_s = w_c = w_w + w_l \; . \tag{19}$$

On the micro-level, there is an identity of the growth rate of intergenerational saving, the growth rate of consumption, and the sum of the growth rates of wage and labor supply. The last sum can also be interpreted as the growth rate of labor income. This identity results from the individuals' budget constraint, i.e. labor income has to equal savings plus consumption in each period

$$w(t)\,l(\theta,t) = s(\theta,t) + c(\theta,t) \; . \tag{20}$$

At first we have to state that only identical preferences of all generations are possible candidates for an equilibrium path. However, when dealing with technical progress there are additional dynamics within the price system and we have to consider the change in the optimal decision between the generations. Does there exist a link between condition (19) and the kind of utility function generating a system of micro-economic growth rates (8)?

Reconsidering the maximization problem of each generation (2) we know

$$\frac{U_{1-l}}{U_c} = w(t) \qquad \forall\, t \in [0,T] \; , \tag{21}$$

which means that the marginal rate of substitution between consumption and labor has to equal the price ratio throughout the whole life-cycle.

Since the wage rate grows at a constant rate, the marginal rate of substitution of all living generations has to follow the respective path. Hence, it is interesting to look at the elasticity of substitution between consumption and labor which determines the development of the optimal decisions of the generations

(22) $$\sigma_{c,1-l} = \frac{\frac{U_{1-l}}{U_c}\, d\left(\frac{c}{1-l}\right)}{\frac{c}{1-l}\, d\left(\frac{U_{1-l}}{U_c}\right)} \;.$$

We substitute (21) and switch to logorithmic derivatives to obtain

(23) $$\sigma_{c,1-l} = \frac{dln\left(\frac{c}{1-l}\right)}{dlnw} = \frac{dlnc - dln\,(1-l)}{dlnw} \;.$$

Written in growth rates, the condition for utility maximizing generations leads to

(24) $$\sigma_{c,1-l} = \frac{w_c - w_{1-l}}{w_w} \;.$$

If all growth rates are constant as the neoclassical theory requires, there is a need for CES-typ utility functions to attain the above properties. Additionally we know that the growth rates of leisure w_{1-l} and labor w_l have to equal zero, because labor is a limited resource and therefore exponential growth will be impossible.

However, equation (24) is only the condition for the marginal utilities, so we have to analyze generation θ's budget constraint

(25) $$\int_{\theta}^{\theta+T} c(\theta,t)e^{-rt}dt - \int_{\theta}^{\theta+T} w(t)l(\theta,t)e^{-rt}dt = 0 \;,$$

too, and obtain

(26) $$e^{w_c\theta}\int_{o}^{T} c(0,t)e^{-rt}dt - e^{(w_w+w_l)\theta}\int_{o}^{T} w(t)l(0,t)e^{-rt}dt = 0$$

by inserting (8).

Hence, an equilibrium can only be supported by the budgets if

(27) $$w_c = w_w \;.$$

Comparing (24) with (27) leads to

(28) $$\sigma_{c,1-l} = 1 \;.$$

Surprisingly, this requires preference functions which are either Cobb-Douglas or log-linear (CES with $\sigma = 1$). Until now economists guessed that homothetic utility functions support a steady state growth path if there are overlapping generations. If we analyze continuous time models, as given above, it is necessary to require CES-type behavior. If additionally labor is an endogenous variable in the model, equilibrium requires for a more restrictive functional form, because leisure is a limited resource and therefore labor supply cannot increase exponentially.

For an intuitive explanation consider individuals with an initial endowment of time which can be used for earning money. The value of this endowment grows at w_w. Since the price for the consumption of leisure grows at the same rate, it is obvious that only Cobb-Douglas or log-linear utility functions support constant labor supply from generation to generation, because the income/wage rate ratio, which determines the supply, is constant over time.

VI. Equilibrium Price System

In the above subsections, we only analyzed the conditions for equilibrium growth rates. Now we investigate the equilibrium price system. Since the real wage rate is growing at w_{Π}, we define the equilibrium price tupel (r, w_o) in such a way that both the labor and the capital market is cleared.

First of all we have to quantify the capital stock $K(t)$. Considering the linear homogeneity of the production function we obtain

$$Y(t) = w(t)L(t) + rK(t) \ . \tag{29}$$

Additionally, the budget identity satisfies

$$Y(t) = C(t) + I(t) = C(t) + \dot{K}(t) \ . \tag{30}$$

Substituting (29) into (30) we get

$$K(t) = \frac{S(t)}{w_K(t) - r} \ . \tag{31}$$

We use (11) and (17) to write the initial capital K_o in a steady state in the following way

$$K_o = \frac{P_o}{w_K - r} \int_o^T e^{-w_K \phi} s(0, \phi) d\phi \ . \tag{32}$$

For an equilibrium analysis it is necessary to calculate the wage/interest-curve from both the demand and supply side.

On the demand side, profit maximizing behavior leads to

$$w_o := \frac{w(t)}{\Pi(t)} = g(\varkappa) - k\,g'(\varkappa) \tag{33}$$

with $\varkappa = \dfrac{K}{\Pi L}$. The function g' is strictly monotonic. Hence there exists an inverse function, such that w_o is a function only depending on the interest rate r[9]

$$w_o = G(r) \text{ with } G'' > 0 . \tag{34}$$

The demand side function (34) therefore is convex. On the other hand the utility maximization and

$$\varkappa_o = \frac{\int_o^T e^{-w_K \phi} s(0,\phi)\,d\phi}{w_o(w_K - r)\int_o^T e^{-w_p \phi} l(0,\phi)\,d\phi} \tag{35}$$

determine the supply side. The optimal saving or labor decision depends on the wage and interest rate. Therefore the fix-points of (35) create the supply side function. Hence, equations (34) and (35) determine the equilibrium E as illustrated in figure 1.[10]

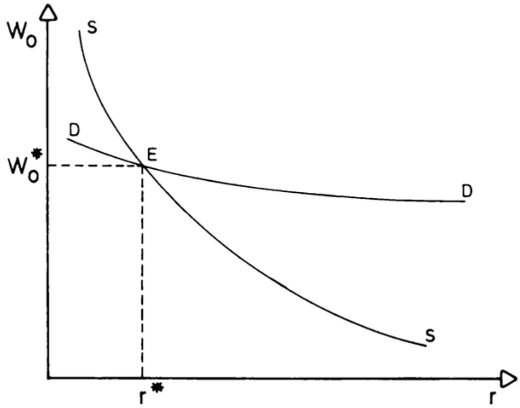

Figure 1

[9] For details see *Krelle* (1985).

[10] *Diamond* (1965) presents a similar argument.

The derivatives of the supply curve have arbitrary signs and thus there may be more than one equilibrium or none. Only by assumption the solution may be unique.

Additionally we have to analyze the sign of the capital stock.

Theorem: If the wealth of each generation is positive within the whole time horizon, then there always is a positive capital stock.

Proof: If $K(t)$ follows (31) and (32), respectively, there are three possible cases:

a) numerator and denominator on the right hand side of (31) and (32) are positive,

b) numerator and denominator are negative,

c) both are zero (which is the case if $w_K = r$ and $S(t) \neq 0$).

Let us start with the last case. We define S_o, the numerator, as follows

(36) $$\psi(w_K) \equiv P_o \int_o^T e^{-w_K \phi} s(0,\phi) d\phi \ ,$$

and write the denominator as

(37) $$\Gamma(w_K) \equiv w_K - r \ .$$

To calculate the fraction for $w_K = r$ we apply the lemma of d'Hospital and obtain

(38) $$K_o = \frac{\psi'}{\Gamma'} = -P_o \int_o^T \phi e^{-w_K \phi} s(0,\phi) d\phi \ .$$

Integration by parts leads to

(39) $$K_o = P_o \left[\int_o^T V(0,\phi) d\phi - [\phi V(0,\phi)]_o^T \right] .$$

The second term of this sum vanishes because there are no bequests. Hence (39) has the same sign as the wealth of the individual. $V(0,\phi)$ cannot change its sign over time because the person always saves first and dissaves later or vice versa.

At last we have to analyze the cases a) and b). K_o is always positive if $S(t)$ or equivalently $\psi(w_K)$ change sign at $w_K = r$. Therefore we have to show $\psi(r) = 0$ and $\psi'(w_K) > 0$. The first is guaranteed by $V(0,T) = 0$, the latter requires

(40) $$\int_o^T V(0,\phi)d\phi > 0 \ .$$

This completes the proof. #

Hence, if positive wealth prevails within an individuals time horizon, net savings of all living generations are positive, too. And thus the economy accumulates a physical capital stock.

VII. The Golden Rule of Accumulation

Phelps, in his 1961 seminal work, characterized an optimal growth path which maximizes consumption over the time horizon. The main result was that capital has to grow with the interest rate, or in other words: the whole labor income has to equal total consumption and the interest payments must be reinvested completely.

Can such an optimal rule be supported by our overlapping generations growth model? At first, we have to state the "golden rule" in growth rates

(41) $$w_Y = w_K = w_p + w_S = r \ .$$

Theorem: If an economy follows the golden rule path, net savings of all generations at each time vanish.

Proof: Equation (4) describes aggregated savings of the economy. Combined with (11) there is

(42) $$S(t) = P_o e^{w_K t} \int_o^T e^{-w_K \phi} s(0,\phi) d\phi \ ,$$

which is equal to zero if $w_K = r$, because in this case the integral gives us the wealth of generation zero at age T. The wealth $V(0,T)$ vanishes because there are no bequests. #

The absence of aggregated net savings from labor income can be explained by the saving and dissaving activities of all generations at the same moment. These activities are balanced in such a way that there is no debt outstanding all the time. Therefore, at the same time all interest revenues are reinvested.

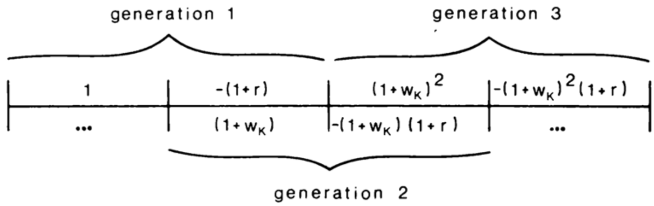

Figure 2

For an intuitive explanation we look at an example: each generation lives two periods, in the first one all individuals save one unit and dissave $1 + r$ units in the second.

If population grows at w_p and saving activities at w_s then,[11] as can be seen in figure 2, the dissavings of the old generation equal the savings of the young and therefore no net-savings of all living generations do exist.

Considering Samuelson[12] this result fits into a pure consumption-loan model of overlapping generations, because in the absence of production $w_{\Pi} = w_S = 0$ and therefore the biological growth rate of the population is equal to the interest factor. Additionally, this expansion path of the economy is optimal, but the reverse of the theorem is not true.

Theorem: If the generations saving function $s(0, t)$ is always zero there is no accumulation of wealth and therefore no capital stock. Hence, this cannot be an equilibrium solution.

Proof: From $s(0, t) = 0$ for all $t \in [0, T]$ it follows

$$V(0, t) = 0 \qquad \forall\, t \in [0, T] \ , \tag{43}$$

and by (31) $K(t) = 0$ for $w_K \neq r$. Additionally for the golden rule $K(t) = 0$ by (39). #

The above theorem can be understood in the following sense. A steady state with no net savings from labor income is either a golden rule path or there is no exchange between generations. In a pure exchange economy both are implementable,[13] but if there is a need for productive capital, the second one gives no solution of our problem.

Theorem: Besides the golden rule and the no exchange path, aggregated net savings from wage income $S(t)$ are positive if $w_K > r$ and negative if $w_K < r$.

Proof: The savings of all living generations can be described by (42) and (36) as

$$S(w_K) \;=\; P_o\, e^{w_K t}\, \psi(w_K) \ . \tag{44}$$

From $S(w_K = r) = 0$ and $\psi'(w_K = r) > 0$ (for $V(0, t) \geq 0$)[14] the proof follows immediately. #

[11] Remember: $w_K = w_p + w_s$.
[12] *Samuelson* (1958).
[13] *Gale* (1973).
[14] See eq. (40) above.

Thus, the theorem is equivalent to

(45) $$\dot{K} \lesseqqgtr rK \quad \text{and} \quad C \gtreqqless wL \quad \text{if} \quad w_K \lesseqqgtr r \; .$$

For $w_K < r$ the interest payments are reinvested only partially. They additionally were used to finance the consumption of all living generations. If $w_K > r$, the capital stock grows more rapidly and wage income is also used to accumulate the physical capital stock.

Finally, let us have a look at the golden rule as a competitive equilibrium. Figure 1 gives an intuitive understanding of the two sides of the market and E characterizes the equilibrium. For w_K is an exogenously determined parameter ($w_K = w_p + w_\Pi$) it is only a matter of chance whether an equilibrium point is golden rule or not.

Summary

In a continuously overlapping generations model with technical progress steady states require Cobb-Douglas of log-linear utility functions if labor is treated explicitly as a decision variable of the individuals. Only in the absence of technical progress the preferences may be arbitrary. The golden rule of accumulation implies an equilibrium where either there is no outstanding debt of all living generations or all individuals do not save. The latter case is not implementable, because without saving capital accumulation is impossible. Even the first case can be unfeasible because the equilibrium price system requires $r \neq w_K$. Therefore the practicability of the golden rule is only a matter of chance.

Zusammenfassung

In einem Modell kontinuierlich überlappender Generationen impliziert der technische Fortschritt, daß nur Cobb-Douglas oder log-lineare Nutzenfunktionen einen stationären Wachstumspfad stützen, wenn das Arbeitsangebot Entscheidungsvariable der Individuen ist. Ohne technischen Fortschritt können die Präferenzen weiterhin beliebig sein. Die goldene Regel der Akkumulation bedingt ein Gleichgewicht, bei dem die Generationen sich entweder nicht verschulden oder nicht sparen. Letzteres ist nicht implementierbar, da ohne Sparen kein Kapital gebildet werden kann. Aber auch der erste Fall kann undurchführbar sein, wenn sich ein Gleichgewichtspreissystem mit $r \neq w_K$ ergibt. Deshalb ist die Durchsetzbarkeit der goldenen Regel nur eine Frage des Zufalls.

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DOI https://doi.org/10.3790/schm.107.4.581 | Generated on 2025-12-18 09:40:33