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On the Valuation and Analysis of Risky Debt: A Theoretical Approach Using a Multivariate Extension of the Merton Model

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Abstract

This paper is concerned with the valuation and analysis of default-risky debt instruments with arbitrary interest and principal payments. For the valuation, we use three nested multivariate extensions of the standard Merton (1974) model for pricing risky zero-coupon bonds. First, we present a valuation framework for pricing single risky debt instruments with arbitrary interest and principal payments. We then extend this framework to enable the valuation of multiple debt instruments issued by the same firm. Finally, we further extend the model to additionally take continuous dividend payments to the equity holders into account. Based on these debt valuation frameworks, we calculate various key figures for the analysis of risky debt from the point of view of risk-neutral and risk-averse investors (e.g., promised and expected yields, default probabilities, recovery rates, distance to default, and expected payments).

Keywords: risky debt valuation, Merton Model

JEL Classification: G12, G21, G31, G32

I. Introduction

In this paper, we present three nested multivariate extensions of the *Merton* (1974) model. The first extension allows for pricing single risky debt instruments with arbitrary interest and repayment structures. We then extend this valuation setup to enable pricing multiple debt instruments issued by the same

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firm. Finally, we present an encompassing valuation model that additionally takes continuous payouts to equity holders into account.

With his seminal paper "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates" (*Merton* 1974) Robert C. Merton laid the foundation for the valuation and analysis of risky debt. The approach has become known as the Merton model and is widely used when analyzing and pricing risky corporate debt. The Merton model applies the insights of the well-known *Black/Scholes* (1973) option pricing theorem to the valuation of corporate debt.[1]

Since the traditional Merton framework is based on a number of limiting assumptions, numerous refinements and extensions have been developed since its publication.[2] We differentiate structural models for the valuation of risky debt based on the assumptions upon which they are based. In the following we distinguish five criteria.[3]

First, we distinguish models based on the assumptions made regarding the risk-free interest rate. On the one hand, there are models in which the risk-free interest is deterministic. However, in reality, the value of risky debt instruments is significantly influenced by interest rate risk. To account for this fact, models with stochastic interest rates have been developed. In these models, a range of different diffusion processes are used to model the interest rates.[4]

Second, we divide structural models with regards to whether or not they take coupon payments and/or interim principal repayments into account. In the simplest case, models assume that the debt to be valued is a zero-coupon bond. Since the face value of the zero-coupon bond must be repaid in full only at maturity, it is assumed that the firm can only default at the debt maturity date. In reality, firms will often have debt instruments with more elaborate interest and repayment modalities. When a firm's debt incorporates interest payments and specific repayment agreements, these interim cash flows must be taken into consideration in the valuation framework. If the firm defaults on a single (interest and/or principal) payment date, all subsequent payments are also defaulted on. Therefore, researchers have developed models for the valuation of debt instruments that exhibit coupon payments (continuous payments or at discrete times) and interim principal repayments. In these models, default can therefore occur before maturity. To be precise, default occurs when the value of the firm's

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[1] We provide a brief review of the basics of the *Merton* (1974) model in the Appendix.

[2] For a survey of the Merton model and its extensions, see *Bohn* (2000) or *Sundaram* and *Das* (2009), for instance.

[3] Other criteria that can be used to distinguish between different models include the incorporation of taxes or profit retention or the use of different stochastic processes for the firm's asset value, for instance.

[4] For a detailed overview of articles that incorporate both default risk and interest rate risk, see *Longstaff/Schwartz* (1995).

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assets falls below a specified threshold. This default threshold (i. e., the bankruptcy trigger, killing price, or lower reorganization boundary) can be determined exogenously or endogenously.[5] In the latter case, the default threshold is optimized by the equity holder.

Third, we classify models depending on whether they assume a finite or infinite maturity for the debt of the firm. In the latter case, the debt is a perpetual bond.

Fourth, we subdivide structural models depending on the number of debt instruments the firm has issued. On the one hand, there are models in which the firm's debt consists of a single debt issue. Other structural models, on the other hand, assume that the firm has issued multiple debt instruments. They do so because in reality firms will often issue different debt instruments. Due to cross-default provisions, the default of one instrument has a direct impact on the other debt instruments in the issuer's portfolio, and these interdependencies influence the valuation of each individual instrument in the portfolio. In models with multiple debt issues the issues can either all have the same maturity or feature different maturities. Furthermore, the issues can be of same seniority or vary with regards to their seniority or subordination.

Fifth, since dividends reduce the value of the firm and thus also have an effect on the price of the risky debt, we differentiate structural models that incorporate dividend payments to equity holders from such that do not.

We can classify the most important structural models according to this framework. The *Merton* (1974) model, for instance, is based on the assumption that the risk-free interest rate is constant. It is assumed that the debt of the firm consists only of a single risky zero-coupon bond with finite maturity. The face value is repaid in full at maturity and the firm can only default at the debt maturity date. The equity is a residual claim and, in the basic model, does not pay dividends.

Black/Cox (1976) also present a structural model with deterministic interest rates. Like *Merton* (1974), they model the firm's debt as a single bond with finite maturity and repayment in full at maturity. However, in their paper, the bond can exhibit coupon payments and default can occur prior to the debt instrument's maturity if the value of the firm's assets falls below an exogenously determined threshold. Furthermore, their model allows taking continuous dividend payments to equity holders into account. *Longstaff/Schwartz* (1995) extend the *Black/Cox* model to incorporate stochastic interest rates.

[5] For a comparison of structural credit risk models with endogenous and exogenous default threshold, we refer the reader to *Imerman* (2013).

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Geske (1977) proposes a model with deterministic interest rates that enables pricing single risky discrete coupon bonds with finite maturity which have to be repaid in full at maturity.[6] The model is based on a technique for valuing compound options and uses the multivariate normal distribution. Default can occur prior to maturity but, as opposed to *Black* and *Cox* (1976), the killing prices are determined endogenously. *Fischer* et al. (2000) present a model based on *Geske* (1977, 1979) that allows for the valuation of debt instruments with partial annual principal repayments.

Leland (1994) also studies the valuation of fixed coupon debt instruments under uncertainty in a framework with constant interest rates. The analysis assumes a single class of debt which must be repaid in full at maturity. However, default can occur before the maturity date if the value of the firm reaches an endogenously determined bankruptcy-triggering condition. Furthermore, the model allows for continuous dividends.

Bao/Hou (2017) extend the Merton model to include multiple zero-coupon bond issues which have the same seniority but different finite maturity dates. The firm can default before the final maturity date if its assets fall below an exogenously determined killing price.

In our paper, we are concerned with multivariate extensions of the *Merton* (1974) model. For simplicity of implementation, we follow *Merton* (1974) in assuming the interest rate is constant (criterion 1) and maturity is finite (criterion 3). The term "risky" is therefore restricted to mean possible gains or losses caused by changes in the probability of default and, in consequence, does not encompass gains or losses resulting from changes in interest rates. However, we relax many of the standard model's limiting assumptions in order to improve the model's usefulness in pricing actual bonds found in the market. First of all, we extend the standard valuation framework to enable pricing risky debt instruments with any kind of interest payment structure and principal repayment agreement (criterion 2). We allow for default prior to maturity and determine the killing prices endogenously following *Geske* (1977). Second, we extend the valuation framework to enable pricing multiple debt instruments within an issuer's debt portfolio. Our model can thus take interdependencies between debt instruments of the same issuer into account (criterion 4). Finally, we extend our valuation framework to allow for continuous dividend payments to equity holders (criterion 5).

Based on our valuation framework, we furthermore illustrate the calculation of key risk and return figures for both risk-neutral and risk-averse investors. To our knowledge, we are the first to provide a framework for pricing multiple debt instruments with arbitrary coupon and principal payments that additionally

[6] See also *Geske* and *Johnson* (1984).

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takes continuous payments to equity holders into account. By including these more realistic assumptions we provide a valuation framework that is more closely aligned with the valuation requirements in practice.

The valuation technique we apply has been very frequently used in modern finance theory since the Nobel Prize-winning paper of *Modigliani/Miller* (1958) with their famous irrelevance proposition theorem. They show that under the conditions of perfect markets (e. g. no taxes, no transactions costs, investors are price takers, securities are freely divisible and trading in the assets takes place continuously) the capital structure has no impact on the value of a firm. The method they apply is to compare the value of a levered firm to the value of an unlevered, but otherwise identical firm. Due to the proof of the irrelevance proposition theorem both firm values have to be identical.

The paper is organized as follows. Section 2 describes the model for valuing single debt instruments with any kind of interest payment structure and repayment agreement. Section 3 extends the model to the valuation of multiple debt instruments within a single firm, highlighting how the presence of multiple instruments influences the valuation of each specific instrument. Section 4 adapts the multiple debt instrument framework to incorporate continuous dividend payments to equity holders. Section 5 concludes.

II. Different Repayment Agreements

The *Merton* (1974) model (for a brief summary see the appendix) is based on the assumption that the firm's debt consists of one single zero-coupon debt instrument. While this assumption simplifies the analysis, it reduces its applicability to valuing actual debt, which often includes more complicated interest and repayment agreements. *Geske* (1977) derives closed-form valuation expressions for determining the value of a firm when the debt takes on the shape of a risky coupon bond. We generalize his coupon bond approach in order to be able to value risky debt for any kind of arrangement regarding the principal repayment (e. g., lump-sum repayment, annuity repayment, constant principal repayment).

1. Valuation Setup

Our considerations in the first model extension are based on a range of assumptions following *Merton* (1974) and *Geske* (1977). The value of the firm (i. e., the total assets), V_T, consists of two classes of claims. On the one hand, the firm has a single, homogenous class of debt in the shape of a single debt instrument with nominal value *Nom* and maturity T. This debt instrument can have any kind of interest and principal payment structure. Each payment to the debtholders is refinanced through new external capital, either using equity from ex-

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isting or new equity holders or using debt. On the other hand, the firm has equity which is seen as a residual claim. The total assets are distributed logarithmically normally and can be described by the diffusion-type stochastic process

$$\frac{dV}{V} = \mu_V \, dt + \sigma_V \, dz \tag{1}$$

where μ_V refers to the drift, and σ_V is the volatility of the return on the firm per unit time, which is assumed to be constant. It is assumed that the term structure is flat, so also the risk-free interest rate r is constant. Further we assume that the conditions of perfect markets are fulfilled and therefore, due to the Modigliani Miller proposition theorem, the value of the levered firm is identical to the value of the unlevered, but otherwise identical firm. Hence, V_t can be interpreted as the value of both the levered and unlevered assets.

Under these assumptions, the outstanding debt at $t=1, \ldots, T-1$ is given by

$$Nom_t = Nom_{t-1} - P_t = Nom - \sum_{\tau=1}^{t} P_\tau \,, \tag{2}$$

where P_t is the proportion of the nominal value repaid at time t, $Nom_T = 0$, and $P_T = Nom_{T-1}$. For a fixed nominal interest rate, i_{nom}, the interest payment in each period $t=1, \ldots, T$ is given by

$$I_t = i_{nom} \cdot Nom_{t-1} \,. \tag{3}$$

In the presence of interest payments and principal repayments throughout the debt instrument's term, the Merton model cannot be used to price the equity or the risky debt of the firm. The equity must be interpreted as a compound call option rather than a simple European call option because the equity holders have two possibilities immediately before each debt payment time t. They can either pay the interest and principal repayments due (which is equivalent to buying a new option), or they can refuse to make the required payments and declare bankruptcy (which is equivalent to letting the option expire worthlessly).

2. Valuation Basics

The value of the equity at maturity T is zero if the interest and principal payments cannot be made. Otherwise, it equals the value of the total assets less the final interest payments and the principal repayments. We can write the firm's equity at maturity as

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(4) $$E_T = \begin{cases} 0 & V_T \le I_T + P_T \\ V_T - (I_T + P_T) & V_T > I_T + P_T \end{cases}$$

The equity is analogous to a call on the value of the firm, V_T, with $I_T + P_T$ as the strike price.

At time $(T-1)^+$, momentarily after the interest and principal payments due in period $T-1$, the value of the equity can be derived using the Black/Scholes formula given by

(5) $$E_{(T-1)^+} = V_{T-1} \cdot N(h_1) - (I_T + P_T) \cdot e^{-r} \cdot N(h_2)$$

where

(6) $$h_1 = \frac{ln \frac{V_{T-1}}{I_T + P_T} + r + \frac{\sigma_V^2}{2}}{\sigma_V},$$

(7) $$h_2 = h_1 - \sigma_V ,$$

and $N(\cdot)$ denotes the standard normal cumulative distribution function.

In consequence, for the value of the risky debt at time $(T-1)^+$ we can write

(8) $$D_{(T-1)^+} = V_{T-1} \cdot (1 - N(h_1)) + (I_T + P_T) \cdot e^{-r} \cdot N(h_2).$$

For the value of the equity just before the payments in period $T-1$ we have

(9) $$E_{(T-1)^-} = \begin{cases} 0 & V_{T-1} \le V_{T-1}^* \\ E_{(T-1)^+} - (I_{T-1} + P_{T-1}) & V_{T-1} > V_{T-1}^* \end{cases}$$

where $E_{(T-1)^+}$ is taken from equation (5).

In order to compare the levered with the unlevered firm we have to assume that all debt payments are refinanced with equity, this implies issuing new equity. Although, of course, this assumption is not at all realistic in practice, we only apply it here as a tricky thought experiment to derive our equity and debt valuations in dependency of the firm's asset values. V_{T-1}^* represents the killing price, or bankruptcy trigger, which is the critical value of the firm at $t = T-1$, where the value of the equity at $(T-1)^-$ is just as large as the interest and principal payments that are due at $t = T-1$. We can write this as

(10) $$E_{T-1}(V_{T-1}^*) = I_{T-1} + P_{T-1}.$$

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The equity holders rationally set this killing price to ensure that the value of the equity remains non-negative immediately after the promised interest and principal payments are made (i.e., at time $(T-1)^{+}$). If the value of the equity would become negative ($V_{T-1} < V_{T-1}^{*}$), the shareholders will not pay interest and principal payments, declare bankruptcy, and transfer the value of the total assets to the debtholders. For $V_{T-1} = V_{T-1}^{*}$ the equity holders are indifferent between their debt payments and the firm's bankruptcy.

One period earlier still, at time $(T-2)^{+}$, the value of the equity is given by

$$\begin{aligned} E_{(T-2)^{+}} &= V_{T-2} \cdot N_2\left(h_1^1, h_1^2; \rho_2\right) - \left(I_T + P_T\right) \cdot e^{-2\cdot r} \cdot N_2\left(h_2^1, h_2^2; \rho_2\right) \\ &\quad - \left(I_{T-1} + P_{T-1}\right) \cdot e^{-r} \cdot N\left(h_2^1\right) \end{aligned} \tag{11}$$

where $N_2(\cdot)$ is the bivariate cumulative standard normal distribution,

$$h_1^1 = \frac{ln\dfrac{V_{T-2}}{V_{T-1}^{*}} + r + \dfrac{\sigma_V^2}{2}}{\sigma_V} \tag{12}$$

$$h_2^1 = h_1^1 - \sigma_V \tag{13}$$

$$h_1^2 = \frac{ln\dfrac{V_{T-2}}{I_T + P_T} + \left(r + \dfrac{\sigma_V^2}{2}\right) \cdot 2}{\sigma_V \cdot \sqrt{2}} \tag{14}$$

$$h_2^2 = h_1^2 - \sigma_V \cdot \sqrt{2}, \tag{15}$$

and

$$\rho_2 = \begin{pmatrix} 1 & \sqrt{1/2} \\ 0 & 1 \end{pmatrix} \tag{16}$$

is the correlation matrix. This approach with multivariate normal distributions and with an endogenously determined killing prices was suggested by *Geske* (1977).

In general, the value of the equity at $(T-s)^{+}$ for $s = 1, \ldots, T$ as a function of the total assets V_{T-s} is given by

$$\begin{aligned} E_{(T-s)^{+}} &= V_{T-s} \cdot N_s\left(h_1^1, \ldots, h_1^s; \rho_s\right) - \sum_{t=0}^{s-1} \left(I_{T-t} + P_{T-t}\right) \cdot e^{-r\cdot(s-t)} \\ &\quad \cdot N_{s-t}\left(h_2^1, \ldots, h_2^{s-t}; \rho_{s-t}\right) \end{aligned} \tag{17}$$

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where $N_s(\cdot)$ is the cumulative standard normal distribution of dimension s, $\tau = 1, \ldots, T$,

(18)
$$h_1^\tau = \frac{\ln\left(\frac{V_{T-s}}{V_{T-s+\tau}^*}\right) + \left(r + \frac{\sigma_V^2}{2}\right) \cdot \tau}{\sigma_V \cdot \sqrt{\tau}}$$

(19)
$$h_2^\tau = h_1^\tau - \sigma_V \cdot \sqrt{\tau},$$

and

(20)
$$\rho_s = \langle \rho_{\tau_1 \tau_2} \rangle = \begin{cases} 1 & \text{if } \tau_1 = \tau_2,\ \tau_1 = 1, \ldots, \tau,\ \tau_2 = 1, \ldots, \tau \\ \sqrt{\tau_1 / \tau_2} & \text{if } \tau_1 < \tau_2, \tau_1 = 1, \ldots, \tau,\ \tau_2 = 1, \ldots, \tau \\ 0 & \text{else.} \end{cases}$$

When $s = T$, equation (17) reflects the value of the equity at $T = 0$.

We determine the killing prices, V_τ^*, following *Fischer, Keber,* and *Maringer* (2000) using a reverse bootstrapping technique from $t = T$ to $t = 1$ by ensuring that the value of the equity immediately after the interest and principal payments fulfills the following condition for each $\tau = 1, \ldots, T$:

(21)
$$E_{\tau^+}\left(V_\tau^*\right) = I_\tau + P_\tau$$

where

(22)
$$V_T^* = I_T + P_T.$$

As mentioned above, the equity can be interpreted as a t-dimensional compound option. This is the case because the equity holders have the option either to pay the interest and principal repayment and buy a $(t-1)$ -dimensional option or to forfeit the firm to the debtholders at each interest payment and repayment date (see also *Geske* (1979)).

Finally, for the value of the risky debt of the firm at $t = 0$ we have

(23)
$$\begin{aligned} D_0 &= V_0 - E_0 \\ &= V_0 \cdot \left[1 - N_T\left(d_1^1, \ldots, d_1^T; \rho_T\right)\right] + \sum_{t=1}^{T} (I_t + P_t) \cdot e^{-r \cdot t} \cdot N_t\left(d_2^1, \ldots, d_2^t; \rho_t\right) \end{aligned}$$ [7]

where

[7] See *Fischer*, *Keber*, and *Maringer* (2000).

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(24)
$$d_1^t = \frac{\ln\left(\frac{V_0}{V_t^*}\right) + \left(r + \frac{\sigma_V^2}{2}\right) \cdot t}{\sigma_V \cdot \sqrt{t}}$$

and

(25)
$$d_2^t = d_1^t - \sigma_V \cdot \sqrt{\tau}.$$

The debtholders can be seen as holders of risk-free debt and writers of a put option on the total assets, while the equity holders can be viewed as the holders of this specific put option. Bankruptcy is analogous to the execution of the option.

3. *Repayment-Specific Formulas*

We present the formulas for valuing a firm's debt for four different types of principal repayment arrangements. First, we take the case that the risky debt is present in the shape of a zero-coupon bond with face value, *Nom*. No interest or principal payments are made during the term of the bond. The only payment occurs at maturity when the face value is paid to the debtholder. In this case, we can determine the value of the risky debt at $t = 0$ using the *Merton* (1974) formula given by

(26)
$$\begin{aligned} D_0 &= V_0 - E_0 \\ &= V_0 \cdot [1 - N(d_1)] + Nom \cdot e^{-r \cdot T} \cdot N(d_2). \end{aligned}$$

Second, we assume the debt of the firm is present in the shape of a coupon bond with lump-sum repayment. This means that interest payments are due periodically according to the time interval specified in the bond indenture (e.g., annually, semi-annually), and that the face value of the debt has to be repaid at maturity. It follows that

(27)
$$I = i_{nom,T} \cdot Nom$$

and for the value of the risky debt at $t = 0$ we have

(28)
$$\begin{aligned} D_0 &= V_0 - E_0 \\ &= V_0 \left[1 - N_T\left(d_1^1, \ldots, d_1^T; \rho_T\right)\right] + \sum_{t=1}^{T} I_t \cdot e^{-r \cdot t} \cdot N_t\left(d_2^1, \ldots, d_2^t; \rho_t\right) \\ &\quad + Nom \cdot e^{-r \cdot T} \cdot N_T\left(d_2^1, \ldots, d_2^T; \rho_T\right).^8 \end{aligned}$$

8 See *Fischer* et al. (2000).

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Third, the specifications of the firm's debt may state annuity repayment. This implies that a constant annuity, which is composed of both interest as well as principal repayments, is due at periodic payment dates. The annuity is defined by

$$Ann = AF_{i_{nom},T} \cdot Nom_0 \tag{29}$$

where

$$AF_{i_{nom},T} = \frac{\left(1+i_{nom,T}\right)^T \cdot i_{nom,T}}{\left(1+i_{nom,T}\right)^T - 1}. \tag{30}$$

In this setting, the interest in each period depends on the outstanding nominal value at the beginning of the corresponding period. We get

$$I_t = i_{nom,T} \cdot Nom_{t-1} \tag{31}$$

where

$$Nom_t = Nom_{t-1} - P_t \tag{32}$$

and

$$\sum_{t=1}^{T} P_t = Nom_0 . \tag{33}$$

The principal repayment at time t is

$$P_t = Ann - I_t = P_1 * \left(1+i_{nom,T}\right)^{(t-1)} \tag{34}$$

and the value of the risky debt at $t=0$ is

$$\begin{aligned} D_0 &= V_0 - E_0 \\ &= V_0 \left[1 - N_T\left(d_1^1,\ldots,d_1^T;\rho_T\right)\right] + \sum_{t=1}^{T} Ann \cdot e^{-r\cdot t} \cdot N_t\left(d_2^1,\ldots,d_2^t;\rho_t\right). \end{aligned} \tag{35}$$

Fourth, the debt may specify constant principal repayments, P. In this case, for the value of the risky debt at $t=0$ we have

$$\begin{aligned} D_0 &= V_0 - E_0 \\ &= V_0 \cdot \left[1 - N_T\left(d_1^1,\ldots,d_1^T;\rho_T\right)\right] + \sum_{t=1}^{T} \left(I_t + P\right) \cdot e^{-r\cdot t} \cdot N_t\left(d_2^1,\ldots,d_2^t;\rho_t\right). \end{aligned} \tag{36}$$ [9]

[9] See *Fischer* et al. (2000).

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In these equations,

$$Prob'(No\ Default\ until\ t) = N_t\left(d_2^1, \dots, d_2^t; \rho_t\right) \tag{37}$$

represents the cumulative risk-neutral survival probability until time t and, conversely,

$$Prob'(Default\ until\ t) = 1 - N_t\left(d_2^1, \dots, d_2^t; \rho_t\right) \tag{38}$$

is the cumulative risk-neutral probability of default (PD) until time t.

4. Interpretation and Analysis

For a more detailed interpretation of the components of the value of the risky debt at time $t = 0$, the generic equation for any kind of repayment specification can be written as

$$\begin{aligned} D_0 &= \sum_{t=1}^{T} (I_t + P_t) \cdot e^{-r \cdot t} \\ &- \sum_{t=1}^{T} \left[(I_t + P_t) \cdot e^{-r \cdot t} - V_0 \cdot \frac{N_{t-1}\left(d_1^1, \dots, d_1^{t-1}; \rho_{t-1}\right) - N_t\left(d_1^1, \dots, d_1^t; \rho_t\right)}{N_{t-1}\left(d_2^1, \dots, d_2^{t-1}; \rho_{t-1}\right) - N_t\left(d_2^1, \dots, d_2^t; \rho_t\right)} \right] \\ &\cdot \left[N_{t-1}\left(d_2^1, \dots, d_2^{t-1}; \rho_{t-1}\right) - N_t\left(d_2^1, \dots, d_2^t; \rho_t\right) \right] \end{aligned} \tag{39}$$

From equation (39), it can be seen that the value of the risky debt can be interpreted as the value of a risk-free bond (first term on the righthand side) less the present value of the expected losses for all periods (summation term). The latter is the sum over the product of two components – the discounted loss given default and the risk-neutral total probability of default – at each period t.

We calculate the risk-neutral total probability of default at time t as

$$\begin{aligned} &Prob'(No\ Default\ until\ t-1\ and\ Default\ at\ t) \\ &= N_{t-1}\left(d_2^1, \dots, d_2^{t-1}; \rho_{t-1}\right) - N_t\left(d_2^1, \dots, d_2^t; \rho_t\right) \\ &= N_{t-1}\left(d_2^1, \dots, d_2^{t-1}, -d_2^t;\ \rho_t^{'}\right) \end{aligned} \tag{40}$$

where

$$N_0 := 1 \tag{41}$$

and

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$$\rho_t' = \langle \rho'_{\tau_1,\tau_2} \rangle = \begin{cases} 1 & \text{if } \tau_1 = \tau_2,\ \tau_1 = 1,\ldots,\ \tau,\ \tau_2 = 1,\ldots,\tau \\ \sqrt{\tau_1/\tau_2} & \text{if } \tau_1 < \tau_2, \tau_1 = 1,\ldots,\tau, \tau_2 = 1,\ldots,\tau - 1 \\ -\sqrt{\tau_1/\tau_2} & \text{if } \tau_2 = \tau, \tau_1 = 1,\ldots,\tau \\ 0 & \text{else} \end{cases} \tag{42}$$ [10]

Equation (39) can therefore be simplified to

$$D_0 = \sum_{t=1}^{T} (I_t + P_t) \cdot e^{-r \cdot t} - \sum_{t=1}^{T} \left[(I_t + P_t) \cdot e^{-r \cdot t} - V_0 \cdot \frac{N_{t-1}\left(d_1^1,\ldots,d_1^{t-1}, -d_1^t;\ \rho_t'\right)}{N_{t-1}\left(d_2^1,\ldots,d_2^{t-1}, -d_2^t;\ \rho_t'\right)} \right] \cdot \left[N_{t-1}\left(d_2^1,\ldots,d_2^{t-1}, -d_2^t;\ \rho_t'\right) \right] \tag{43}$$

Besides the cumulative and the total risk-neutral default probabilities, we can also determine the conditional risk-neutral probability of default using

$$\begin{aligned} &Prob'(Default\ at\ t \mid No\ Default\ until\ t-1) = \\ &= \frac{Prob'(Default\ until\ t) - Prob'(Default\ until\ t-1)}{Prob'(No\ Default\ until\ t-1)} \\ &= \frac{N_{t-1}\left(d_2^1,\ldots,d_2^{t-1}; \rho_{t-1}\right) - N_t\left(d_2^1,\ldots,d_2^t; \rho_t\right)}{N_{t-1}\left(d_2^1,\ldots,d_2^{t-1}; \rho_{t-1}\right)}. \end{aligned} \tag{44}$$

Using these insights, the risk-neutral recovery rate equals

$$\begin{aligned} RR_t' &= \frac{V_0 \cdot e^{r \cdot t}}{I_t + Nom_{t-1}} \cdot \frac{N_{t-1}\left(d_1^1,\ldots,d_1^{t-1}; \rho_{t-1}\right) - N_t\left(d_1^1,\ldots,d_1^t; \rho_t\right)}{N_{t-1}\left(d_2^1,\ldots,d_2^{t-1}; \rho_{t-1}\right) - N_t\left(d_2^1,\ldots,d_2^t; \rho_t\right)} \\ &= \frac{V_0 \cdot e^{r \cdot t}}{I_t + Nom_{t-1}} \cdot \frac{N_{t-1}\left(d_1^1,\ldots,d_1^{t-1}, -d_1^t;\ \rho_t'\right)}{N_{t-1}\left(d_2^1,\ldots,d_2^{t-1}, -d_2^t;\ \rho_t'\right)}. \end{aligned} \tag{45}$$

We derive the continuous (expected) yield to maturity of debt for risk-neutral investors, $E_0'(y_T)$, from

$$D_0 = \sum\nolimits_{t-1}^{T} E_0'\left(Cash\ Flow_t\right) \cdot e^{-E_0'(y_T) \cdot t} \tag{46}$$

using the expected risk-neutral cash flow for each period. We can write the expected risk-neutral cash flow as

[10] For the proof see *Fischer* et al. (2000), Appendix A.

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$$E_0'(Cash\ Flow_t) = (I_t + P_t) \cdot Prob'(No\ Default\ until\ t) + E_0'(V_t \mid No\ Default\ until\ t-1\ and\ Default\ at\ t) \tag{47}$$

where

$$\begin{aligned} &E_0'(V_t \mid No\ Default\ until\ t-1\ and\ Default\ at\ t) \\ &= RR' \cdot (Nom_{t-1} + I_t) \cdot \left[N_{t-1}\left(d_2^1, \ldots, d_2^{t-1}; \rho_{t-1}\right) - N_t\left(d_2^1, \ldots, d_2^t; \rho_t\right)\right] \\ &= V_0 \cdot e^{r \cdot t} \cdot \left[N_{t-1}\left(d_1^1, \ldots, d_1^{t-1}; \rho_{t-1}\right) - N_t\left(d_1^1, \ldots, d_1^t; \rho_t\right)\right]. \end{aligned} \tag{48}$$

Consequently, for the value of the debt at $t=0$ we have

$$\begin{aligned} D_0 = \sum\nolimits_{t-1}^{T} \{ &V_0 \cdot e^{r \cdot t} \cdot \left[N_{t-1}\left(d_1^1, \ldots, d_1^{t-1}; \rho_{t-1}\right) - N_t\left(d_1^1, \ldots, d_1^t; \rho_t\right)\right] \\ &+ (I_t + P_t) \cdot N_t\left(d_2^1, \ldots, d_2^t; \rho_t\right)\} \cdot e^{-E_0'(y_T) \cdot t}. \end{aligned} \tag{49}$$

5. *Risk Aversion*

It has been shown that, in reality, the assumption of risk-neutral investors, which is often made in theoretical models, rarely holds. Typical investors are risk-averse and not willing to invest at the risk-free interest rate. Instead, they require compensation for bearing risk and are therefore more interested in the risk-adjusted yield rather than the risk-neutral yield. We additionally outline the calculation of the risk-averse probabilities as well as the risk-adjusted yield to cater to this preference.

In the risk-neutral setting, the risk-free interest rate t is used to calculate the risk-neutral yield based on the promised interest and principal payments. In the risk-averse setting, r alone can no longer be used. Instead, the instantaneous drift of the total assets, μ_V, must be calculated in order to determine the risk-adjusted yield based on the risk-averse expected interest and principal payments. This is done based on the intertemporal CAPM by *Merton* (1973a) given by

$$\mu_V = r + (\mu_M - r) \cdot \beta_V \tag{50}$$

where μ_M is the drift of the market of unlevered assets, and β_V is the beta factor of the firm's assets. The market drift is given exogenously, whereas the asset beta factor can either be exogenously given or determined iteratively from

$$\beta_E = \Delta_E \cdot \frac{V_0}{E_0} \cdot \beta_V \tag{51}$$

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where

$$\Delta_E = N_T\left(d_1^1,\ldots,d_1^T;\rho_T\right) \tag{52}$$

and the equity beta β_E (instantaneous systematic risk of equity) is estimated empirically. We calculate the risk-averse survival probability until time t using μ_V such that

$$Prob(No\ Default\ until\ t) = N_t\left(k_2^1,\ldots,k_2^t;\rho_t\right) \tag{53}$$

where

$$k_1^\tau = \frac{\ln\left(\frac{V_0}{V_\tau^*}\right) + \left(\mu_V + \frac{\sigma_V^2}{2}\right)\cdot\tau}{\sigma_V\cdot\sqrt{\tau}} \tag{54}$$

and

$$k_2^\tau = k_1^\tau - \sigma_V\cdot\sqrt{\tau}. \tag{55}$$

Conversely, for the risk-averse cumulative default probability until time t we have

$$Prob(Default\ until\ t) = 1 - N_t\left(k_2,\ldots,k_2;\ \ _t\right). \tag{56}$$

The risk-averse total default probability at time t is

$$\begin{aligned} &Prob(No\ Default\ until\ t-1\ and\ Default\ at\ t) \\ &= N_{t-1}\left(k_2^1,\ldots,k_2^{t-1};\rho_{t-1}\right) - N_t\left(k_2^1,\ldots,k_2^t;\rho_t\right) \\ &= N_{t-1}\left(k_2^1,\ldots,k_2^{t-1},-k_2^t;\ \rho_t^{'}\right). \end{aligned} \tag{57}$$

Finally, for the conditional risk-averse probability of default at time t we get

$$\begin{aligned} &Prob(Default\ at\ t\,|\,No\ Default\ until\ t-1) = \\ &= \frac{Prob(Default\ until\ t) - Prob(Default\ until\ t-1)}{Prob(No\ Default\ until\ t-1)}. \\ &= \frac{N_{t-1}\left(k_2^1,\ldots,k_2^{t-1};\rho_{t-1}\right) - N_t\left(k_2^1,\ldots,k_2^t;\rho_t\right)}{N_{t-1}\left(k_2^1,\ldots,k_2^{t-1};\rho_{t-1}\right)}. \end{aligned} \tag{58}$$

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The risk-averse recovery rate is

(59) $$RR_t = \frac{V_0 \cdot e^{\mu_V \cdot t}}{I_t + Nom_{t-1}} \cdot \frac{N_{t-1}\left(k_1^1,\ldots,k_1^{t-1};\rho_{t-1}\right) - N_t\left(k_1^1,\ldots,k_1^t;\rho_t\right)}{N_{t-1}\left(k_2^1,\ldots,k_2^{t-1};\rho_{t-1}\right) - N_t\left(k_2^1,\ldots,k_2^t;\rho_t\right)}..$$

We derive the continuous (expected) yield to maturity of debt for risk-averse investors, $E_0\left(y_T\right)$, from

(60) $$D_0 = \sum\nolimits_{t01}^{T} E_0\left(Cash\ Flow_t\right) \cdot e^{-E_0(y_T)\cdot t}$$

here the expected cash flows at each payment date are given by

(61) $$\begin{aligned} E_0\left(Cash\ Flow_t\right) &= \left(I_t + P_t\right) \cdot N_t\left(k_2^1,\ldots,k_2^t;\rho_t\right) \\ &\quad + V_0 \cdot e^{\mu_V \cdot t} \cdot \left[N_{t-1}\left(k_2^1,\ldots,k_2^{t-1};\rho_{t-1}\right) - N_t\left(k_2^1,\ldots,k_2^t;\rho_t\right)\right]. \end{aligned}$$

It follows that the market value of debt at $t=0$ is

(62) $$D_0 = \sum\nolimits_{t-1}^{T} \left\{ \begin{matrix} V_0 \cdot e^{\mu_V \cdot t} \cdot \left[N_{t-1}\left(k_1^1,\ldots,k_1^{t-1};\rho_{t-1}\right) - N_t\left(k_1^1,\ldots,k_1^t;\rho_t\right)\right] \\ + \left(I_t + P_t\right) \cdot N_t\left(k_2^1,\ldots,k_2^t;\rho_t\right) \end{matrix} \right\} \cdot e^{-E_0(y_T)\cdot t}$$

It is then possible to calculate the instantaneous volatility of debt, σ_D, and equity, σ_E,

(63) $$\sigma_D = \Delta_D \cdot \frac{V_0}{D_0} \cdot \sigma_V$$

(64) $$\sigma_E = \Delta_E \cdot \frac{V_0}{E_0} \cdot \sigma_V$$

Where

(65) $$\Delta_D = \left[1 - N_T\left(d_1^1,\ldots,d_1^T;\rho_T\right)\right]$$

and

(66) $$\Delta_E = N_T\left(d_1^1,\ldots,d_1^T;\rho_T\right)$$

as well as the debt (instantaneous systematic risk of debt) and equity betas

(67) $$\beta_D = \Delta_D \cdot \frac{V_0}{D_0} \cdot \beta_V$$

(68) $$\beta_E = \Delta_E \cdot \frac{V_0}{E_0} \cdot \beta_{V.}$$

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Furthermore, it may be of interest to calculate the distance to default. The distance to default shows how many standard deviations of the return of the asset lie between the value of the asset and its bankruptcy point (i. e., the killing price, V_t^*). We can write this as

(69) $$N_T\left(DD_1',\ldots,DD_T';\rho_T\right)=N_T\left(d_2^1,\ldots,d_2^T;\rho_T\right).$$

Since the risk-neutral survival probability is given by

(70) $$Prob'(Survive\ until\ t)=N_t\left(d_2^1,\ldots,d_2^t;\rho_t\right),$$

we can write the distance to default for risk-neutral investors for each period t, DD_t', as

(71) $$DD_t' = \frac{ln\frac{V_0}{V_t^*}+\left(r-\frac{\sigma_V^2}{2}\right)t}{\sigma_V\sqrt{t}}.$$

For risk-averse investors, the derivation applies analogously. The risk-averse distance to default is given by

(72) $$N_T\left(DD_1,\ldots,DD_T;\rho_T\right)=N_T\left(k_2^1,\ldots,k_2^T;\rho_T\right).$$

Again, since the risk-averse survival probability is

(73) $$Prob(Survive\ until\ t)=N_t\left(k_2^1,\ldots,k_2^t;\rho_t\right),$$

for the risk-adjusted distance to default for each period t, DD_t we have

(74) $$DD_t = \frac{ln\frac{V_0}{V_t^*}+\left(\mu_V-\frac{\sigma_V^2}{2}\right)t}{\sigma_V\sqrt{t}}.$$

6. *Numerical Example*

We present a numerical example to illustrate our formulas for the coherent valuation of the debt. We value the debt of a firm whose total assets consist of non-dividend paying equity and a single debt instrument, namely a loan with lump-sum repayment. The basic parameters are shown in Table 1. The value of the total assets of the firm amounts to 100, the volatility of the assets is 15 %, and the beta of the firm's assets is 1. The nominal value of the debt instrument is 70, and the maturity of the debt is five years. Furthermore, the risk-free rate of interest is 2 % p. a. and the drift of the market of unlevered assets is 4 % p. a.

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Table 1

Parameters of a Single Debt Instruments

This table reports the parameters used to illustrate the valuation of risky debt in the numerical examples.

Parameter	Value
Term T	5
Asset value V_0	100.00
Nominal value Nom_0	70.00
Risk-free rate r	2.00 %
Asset volatility σ_V	15.00 %
Asset beta β_V	1.00
Drift of the market of unlevered assets μ_M	4.00 %

As can be seen from Table 2, the firm pays annual interest at a nominal interest rate of 2.5 % on its loan. The value of the risk-free debt at $t=0$, D_0^{rf}, which is the present value of the promised payments discounted with the risk-free interest rate, amounts to 70.58. This corresponds to the first term on the righthand side of equation (39). The value of the risky debt at $t=0$, D_0, is 70.24.[11] As mentioned in Section 2.4, this corresponds to the value of the risk-free debt less the present value of the expected losses for all future periods.

[11] We perform our calculations in the statistical computing software R (www.r-project.org) using the package *mvtnorm*, specifically the command *pmvnorm,* to calculate the distribution function of the multivariate normal distribution.

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Table 2

Valuation Results for Firm with Single Debt Instrument (Lump-Sum)

This table presents the valuation results for a firm with a single debt instrument and non-dividend paying equity based on the parameters specified in Table 1. The debt instrument is a lump-sum loan with a nominal interest rate of 2.5 % p. a. The value of the risk-free debt, D_0^{rf}, is derived as the present value of the promised payments, while the value of the risky debt, D_0, is the present value of the expected payments. Regarding the cumulative, total, and conditional probabilities of default, the recovery rates, the expected cash flows as well as the distance to default, parameters indicated with a dash refer to risk-neutral results. Parameters without a dash are results derived using the risk-averse approach.

	Time t					
	0	1	2	3	4	5
Value of risk-free debt D_0^{rf}	70.58					
Value of risky debt D_0	70.24					
Killing prices V_t^*		60.08	60.91	62.18	64.45	71.75
Cumulative PD_t'		0.03 %	0.79 %	2.95 %	6.51 %	14.17 %
Cumulative PD_t		0.02 %	0.46 %	1.70 %	3.80 %	8.56 %
Total PD_t'		0.03 %	0.76 %	2.16 %	3.56 %	7.66 %
Total PD_t		0.02 %	0.45 %	1.24 %	2.10 %	4.75 %
Conditional PD_t'		0.03 %	0.76 %	2.18 %	3.67 %	8.19 %
Conditional PD_t		0.02 %	0.45 %	1.25 %	2.13 %	4.94 %
Recovery rate RR'		80.65 %	79.42 %	78.14 %	83.58 %	89.57 %
Recovery rate RR_t		80.74 %	79.67 %	80.27 %	81.90 %	91.71 %
$E_0'(Cash\ Flow_t)$		1.77	2.17	2.91	3.77	66.51
$E_0(Cash\ Flow_t)$		1.76	2.00	2.43	2.92	68.74
Distance to default DD_t'		3.46	2.42	1.93	1.58	1.12
Distance to default DD_t		3.59	2.61	2.16	1.85	1.42

Table 2 also gives an overview of the killing prices for each period, which are monotonically increasing. While the cumulative probabilities of the default (*PD*) increase over time by definition, it can be seen that also the total, as well as the

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conditional probabilities, increase over time. Both the risk-neutral as well as the risk-averse default probabilities increase more strongly with longer maturities. The risk-averse probabilities are uniformly lower than their risk-neutral counterparts. The recovery rates (*RR*) also increase towards maturity. However, the risk-adjusted recovery rates exceed the risk-neutral rates in all periods. Table 2 also shows the expected risk-adjusted cash flows in comparison with the expected risk-neutral cash flows for each period. The risk-neutral cash flows exceed their risk-adjusted counterparts in all periods except period $t=5$, where the repayment of the face value is expected to take place. The distance to default (*DD*) for the lump-sum loan decreases with time, with the risk-averse distance exceeding the risk-neutral values.

Next, we compare the valuation results for the lump-sum loan with three other scenarios. We again value the same firm as mentioned above, using the parameters specified in Table 1. For each valuation, we replace the firm's debt with a different debt instrument. The nominal value of the debt in each scenario remains 70, merely the interest payment structure and the repayment modalities change. The results are summarized in Table 3. The first numerical column shows the results for the firm with the lump-sum loan, which we valued at the beginning of this section. Column 2 shows the results when the loan is equipped with annuity repayment instead. Column 3 depicts the case where constant principal repayments are specified for the debt instruments. Finally, column 4 shows the results when the debt instrument is present in the shape of a zero-bond that is redeemed at its nominal value at maturity. As can be seen from Table 3, in the first three scenarios, the debt pays annual interest of 2.5 %. The zero-bond naturally does not pay any interest.

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Table 3

Valuation Results for Firm with Single Debt Instrument (Different Repayment Scenarios)

This table reports the valuation results for a firm with a single debt instrument and non-dividend paying equity based on the parameters specified in Table 1. The four numeric columns correspond to four valuation scenarios. In each scenario, the firm's single debt instrument has a different interest and repayment structure. The lump-sum, annuity, and constant principal loans all feature a nominal interest rate of 2.5 % p. a. The zero-bond naturally does not pay any periodic interest. The value of the risk-free debt, D_0^{rf}, is derived as the present value of the promised payments, while the value of the risky debt, D_0, is the present value of the expected payments.

	Form of Repayment			
	Lump-Sum	Annuity	Constant Principal	Zero-Coupon
Value of risk-free debt D_0^{rf}	70.58	70.98	70.96	63.34
Value of risky debt D_0	70.24	70.92	70.91	62.29
Instantaneous debt volatility σ_D	1.71 %	0.21 %	0.21 %	1.68 %
Instantaneous equity volatility σ_E	46.36 %	51.07 %	51.06 %	37.00 %
Debt beta β_D	0.11	0.01	0.01	0.11
Equity beta β_E	3.09	3.40	3.40	2.47
Debt drift μ_D	2.23 %	2.03 %	2.03 %	2.22 %
Equity drift μ_D	8.18 %	8.81 %	8.81 %	6.93 %
Promised continuous yield to maturity y_T	2.40 %	1.87 %	2.03 %	2.33 %
Expected risk-neutral continuous yield to maturity $E_0'(y_T)$	2.00 %	2.00 %	2.00 %	2.00 %
Expected risk-averse continuous yield to maturity $E_0(y_T)$	2.17 %	2.01 %	2.01 %	2.17 %

Again, the value of the risk-free and risky debt instrument is calculated for each scenario. For the lump-sum loan, the annuity loan, and the loan with constant principal repayments, this is done using the formulas presented in Section 2.3. The present value of the zero-coupon debt is derived using the *Merton* (1974) formula (equation (26)).

The results show that the instantaneous volatility of debt is much higher for lump-sum repayment and the zero-coupon bond compared to annuity and con-

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stant principal repayment. This is the case because the repayment of both debt instruments occurs late in the debt instruments' lifetime. The promised continuous yield to maturity is the maximum yield that can be achieved via these debt instruments. Naturally, the expected continuous yields to maturity lie below the promised yields, and the expected risk-neutral yield corresponds to the risk-free interest rate for each debt instrument. The risk-averse expected continuous yields to maturity lie below their promised counterparts but above the expected risk-neutral yields. They are highest for the lump-sum and zero-coupon scenarios.

III. Multiple Debt Instruments

In many cases, a firm will issue not only a single debt instrument but multiple debt instruments. When a firm's debt consists of a portfolio of different instruments, the valuation of each specific debt instrument needs to be modified since the recovery rates in the event of bankruptcy change at each point in time. The valuation of the equity and the determination of the trigger values for bankruptcy do not change and can be carried out as described in the previous section.

1. Valuation Setup

In our second extension of the basic model, we assume the firm has multiple debt instruments in addition to equity. We again base our formulas on a few simple assumptions. First, it is assumed that all debt securities will mature at time T and that they are all of the same rank. Furthermore, the nominal value of the entire debt at time t, Nom_t, is the sum of the specific debt instrument under consideration, Nom_t^S, and the remaining miscellaneous debt instruments, Nom_t^M.

$$Nom_t = Nom_t^S + Nom_t^M \tag{75}$$

The total interest and principal payments are, in turn, the sum of the respective specific and miscellaneous parts.

$$I_t = I_t^S + I_t^M \tag{76}$$

$$P_t = P_t^S + P_t^M \tag{77}$$

In consequence, we can write the claims of the creditors of the specific debt capital to be valued at time t as

$$\gamma_t = \frac{Nom_{t-1}^S + I_t^S}{Nom_{t-1} + I_t}. \tag{78}$$

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2. *Valuation*

For the value of a specific debt instrument at maturity we have

$$D_T^S = \begin{Bmatrix} \gamma_T \cdot V_T \ \ if \ \ V_T < Nom_{T-1}^S + I_T^S \\ Nom_{T-1}^S + I_T^S \ \ if \ \ V_T \geq Nom_{T-1}^S + I_T^S \end{Bmatrix}. \tag{79}$$

At time $(T-1)^-$, momentarily before the final interest and principal payments, we can express the value of the debt as

$$\begin{aligned} D^S_{(T-1)^-} &= \begin{Bmatrix} \gamma_{T-1} \cdot V_{T-1} \ \ if \ \ V_{T-1} < V^*_{T-1} \\ 0 \ if \ \ V_{T-1} \geq V^*_{T-1} \end{Bmatrix} \\ &+ \begin{Bmatrix} 0 \ if \ V_{T-1} < V^*_{T-1} \\ \gamma_T \cdot V_{T-1} \ \ if \ V_{T-1} \geq V^*_{T-1} \end{Bmatrix} \\ &- \begin{Bmatrix} 0 & if \ V_{T-1} < V^*_{T-1} \\ \gamma_T \cdot E_{(T-1)^+} - (P^S_{T-1} + I^S_{T-1}) & if \ V_{T-1} \geq V^*_{T-1} \end{Bmatrix}. \end{aligned} \tag{80}$$

The first term and the middle term on the righthand side of equation (80) can be seen as a short put and a long call, respectively, on γ_{T-1} parts of the total assets against a payment of zero. The third term is a short call on γ_T parts of the equity against a payment of $P^S_{T-1} + I^S_{T-1}$. This short call is virtually a compound option since the equity itself represents an option on the total assets.

One period earlier still, at time $(T-2)^+$, momentarily after the interest and principal payments, the value of the debt is

$$\begin{aligned} D^S_{(T-2)^+} &= V_{T-2} \cdot \left[\gamma_{T-1} + (\gamma_T - \gamma_{T-1}) \cdot N(h_1^1) - \gamma_T \cdot N_2(h_1^1, h_1^2; \rho_2)\right] \\ &+ \left(I^S_{T-1} + P^S_{T-1}\right) \cdot e^{-r} \cdot N(h_2^1) + \left(I^S_T + P^S_T\right) \cdot e^{-2 \cdot r} \cdot N_2(h_2^1, h_2^2; \rho_2) \end{aligned} \tag{81}$$

where

$$h_1^1 = \frac{\ln\left(\frac{V_{T-2}}{V^*_{T-1}}\right) + r + \frac{\sigma_V^2}{2}}{\sigma_V} \tag{82}$$

$$h_2^1 = h_1^1 - \sigma_V \tag{83}$$

and

$$h_1^2 = \frac{\ln\left(\frac{V_{T-2}}{I^S_T + P^S_T}\right) + \left(r + \frac{\sigma_V^2}{2}\right) \cdot 2}{\sigma_V \cdot \sqrt{2}} \tag{84}$$

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$$h_2^2 = h_1^2 - \sigma_V \cdot \sqrt{2}. \tag{85}$$

Analogous to the value of the equity, we calculate the market value of the specific debt instrument at $t=0$ as

$$D_0^S = V_0 \sum_{t=1}^{T-1} \gamma_t \cdot N_t \left(d_1^1, \ldots, d_1^{t-1}, -d_1^t; \rho_t'\right) + \sum_{t=1}^{T} \left(I_t^S + P_t^S\right) e^{-r \cdot t} \cdot N_t \left(d_2^1, \ldots, d_2^t; \rho_t\right). \tag{86}$$

In the special case that γ_t is constant, we can write the market value of the specific debt as

$$D_0^S = \gamma \cdot V_0 \left[1 - N_T \left(d_1^1, \ldots, d_1^T; \rho_T\right)\right] + \sum_{t=1}^{T} \left(I_t^S + P_t^S\right) e^{-r \cdot t} \cdot N_t \left(d_2^1, \ldots, d_2^t; \rho_t\right). \tag{87}$$

3. *Default Clause Regulations*

Bondholders may call their bonds prematurely if the firm fails to service interest or principal payments on time. When a firm has multiple debt instruments outstanding, bondholders may even call their bonds prematurely if the firm fails to pay interest or principal repayments on time on any of the *other* debt instruments it has issued. This is regulated under the international default clause, which grants creditors the right to demand the immediate repayment of their bond or loan amount outstanding as soon as the debtor shows certain signs of a potential default (e.g., insolvency).

Similar regulations can be found under Anglo-American law in the so-called cross-default clause. This is a clause in loan agreements and bond indentures according to which the default of a debtor in another credit relationship entitles the creditor to demand the early termination of his own obligation towards that debtor without there being any *direct* reason for termination. This is often referred to as acceleration.

4. *Practical Implementation*

In practical implementations, the data for the valuation of the different debt instruments is obtained from a range of sources. For example, data is drawn from the plan balance sheets and the plan profit and loss statements, as well as the firm's interest and redemption schedule. Data from the firm's strategic investment plan and its financial plan is also used in order to ensure that planned future borrowings can be taken into account. In addition to the firm-specific information, forecasts of future interest rate levels play an important role.

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The procedure for practically implementing the calculations presented above is straightforward. When the present value of the assets, V_0, and the volatility of the assets, σ_V, are known, first, the interest and redemption schedules for all debt instruments are drawn up. From these schedules, the claims of each creditor of the specific debt instrument to be priced at time t, γ_t, can be calculated. The equity capital is priced recursively from $T-1$, $T-2$, …, 1, 0, and the killing prices V_t^* for each case are determined using the reverse bootstrapping procedure mentioned in Section 2.2. Finally, each specific debt instrument can be valued at $t=0$ using the formulas presented above.

The more realistic situation is that V_0 and σ_V are unknown. In this case, V_0 and σ_V must be calibrated from the present value of the equity and its corresponding volatility, which are known for listed firms. In the first step, the interest and redemption schedules for all debt instruments must be drawn up. Next, the reverse bootstrapping procedure is carried out to recursively value the equity capital from $T-1$, $T-2$, …, 1, 0, and to determine the killing prices V_t^* for each period.[12] This simultaneously leads to V_t^*, V_0, and σ_V. Finally, each specific debt instrument and the total debt capital can be valued at $t=0$ using the formulas presented above.

5. *Numerical Example*

We again present a numerical example to illustrate our formulas for the valuation of specific debt instruments in a firm's debt portfolio. We build on the example presented in the previous section. Recall that we analyzed one firm with one single debt instrument in different interest and repayment modality scenarios (i.e., lump-sum, annuity, constant principal, zero-coupon). Here, we value one single firm but make the assumption that the firm's assets consist of two debt instruments in addition to its non-dividend paying equity. To be precise, the debt portfolio of the firm under consideration consists of a lump-sum loan and a zero-coupon bond. The parameters from Table 1 are still valid; merely the total asset value is now changed to 200 as can be seen from Table 4. Each debt instrument has a nominal value of 70. This ensures that the debt-to-asset ratio at $t=0$ is identical to that of the previous examples. The lump-sum loan again pays annual interest of 2.5 % while the zero-coupon debt pays no interest. Both loans are redeemed at par at maturity.

[12] This approach was first employed within Moody's KMV model, a structural default prediction model frequently used in practice. For a detailed description of the derivation, see *Saunders* and *Allen* (2010).

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Table 4

Valuation Results for Firm with Two Debt Instruments (Lump-Sum & Zero-Coupon)

This table reports the valuation results for a firm whose debt consists of two debt instruments, namely a lump-sum loan and a zero-coupon loan, and non-dividend paying equity. The value of the total assets at time $t=0$ amounts to 200. Each specific debt instrument has a nominal value of 70, and the lump-sum loan features a nominal interest rate of 2.5 % p. a. The zero-bond naturally does not pay any periodic interest. The value of the risk-free debt, D_0^{rf}, is derived as the present value of the promised payments while the value of the risky debt, D_0, is the present value of the expected payments.

	Form of Repayment	
	Lump-Sum	Zero-Coupon
Share on total debt $\gamma_t = \gamma$	50.62 %	49.38 %
Value of risk-free debt D_0^{rf}	70.58	63.34
Value of risky debt D_0	70.35	62.23
Instantaneous debt volatility σ_D	2.98 %	3.37 %
Instantaneous equity volatility σ_E	41.39 %	41.39 %
Debt beta β_D	0.20	0.22
Equity beta β_E	2.76	2.76
Debt drift μ_D	2.40 %	2.45 %
Equity drift μ_E	7.52 %	7.52 %
Promised continuous yield to maturity y_T	2.37 %	2.35 %
Expected risk-neutral continuous yield to maturity $E_0'(y_T)$	2.00 %	2.00 %
Expected risk-averse continuous yield to maturity $E_0(y_T)$	2.17 %	2.16 %

We value each specific debt instrument using the valuation formulas from Section 3.2. The results are given in Table 4. The central revelation from the valuation can be taken from the second numerical column, which contains the results for the zero-coupon bond. Take the value of the risk-free debt and the value of the risky debt, which are 63.32 and 62.23, respectively. Recall the valuation results for the zero-coupon debt in the single debt instrument case in Section 2.6. The value of the risk-free debt in the single debt instrument case was 63.34, which is the same as here in the multiple debt instrument scenario. The value of

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the risky debt, however, was 62.29 in the single debt scenario compared to 62.23 in the multiple debt scenario. This underlines that the standard Merton formula can no longer be used to value zero-coupon debt in situations where a firm's debt consists of a portfolio of different debt instruments. This is due to the fact that, as mentioned in Section 3.3, the default of one instrument in a debt portfolio has a direct impact on the other debt instruments in the portfolio and thus diminishes the value of the other instruments. This interdependency is also reflected in the expected risk-averse continuous yield to maturity, which is 2.16 % for the zero-coupon debt in the multi-debt scenario and 2.17 % in the single-debt scenario.

Table 4 also shows that the structure of the debt is irrelevant for the valuation of the equity. The instantaneous equity volatility, the equity beta, as well as the equity drift, are identical for both the lump-sum and the zero-coupon debt.

IV. Multiple Debt Instruments with Continuous Dividends

The previous sections were based on the assumption that the firm pays no dividends to its equity holders. In our third and final extension of the basic Merton model, we extend the previous considerations to incorporate continuous dividends. These dividends have a constant return of q.[13] They reduce the value of the firm and are thus accounted for in the stochastic process of the firm's value via

$$\frac{dV_t}{V_t} = (\mu_V - q)dt + \sigma_V dz. \tag{88}$$

The equity holders receive continuous dividend payments of $q \cdot V_t$ from $t=0$ until bankruptcy or $t=T$, whichever comes first. The adjustments of the Black/Scholes option pricing formulas for continuous dividends were first derived in *Merton* (1973b).

1. Valuation

As shown by *Galai/Wiener* (2015), the present value of the expected continuous dividends under the classical Merton model is

[13] These assumed continuous dividend payments follow a simple rule, but may not be optimal in some some sense. For the vast literature in this context see *Demarzo* and *Sannikov* (2006) and *Décamps* et al. (2011).

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(89)
$$
\begin{aligned}
VD_0 &= \int_0^T E_0'(Div_t)e^{-rt}\,dt \\
&= \int_0^T q\cdot E_0'(V_\tau)e^{-rt}\,dt = \int_0^T q\cdot V_0\cdot e^{(r-q)\cdot t}\cdot e^{-rt}\,dt\,. \\
&= \int_0^T q\cdot V_0\cdot e^{-q\cdot t}\,dt = V_0\cdot\left(1-e^{-q\cdot T}\right)
\end{aligned}
$$

and the ex-dividend value of the total assets is

(90)
$$V_0^{ex} = V_0 - VD_0 = V_0 - V_0\cdot\left(1-e^{-q\cdot T}\right) = V_0\cdot e^{-q\cdot T}\,.$$

When the firm's debt is present in the shape of a zero-bond, we can calculate the value of the risky debt as

(91)
$$D_0 = V_0^{ex}\cdot\left[1-N\left(d_1^{ex}\right)\right] + Nom\cdot e^{-r\cdot T}\cdot N\left(d_2^{ex}\right)$$

where

(92)
$$d_1^{ex} = \frac{ln\dfrac{V_0}{Nom} + \left(r-q+\dfrac{\sigma_V^2}{2}\right)T}{\sigma_V\cdot\sqrt{T}}$$

and

(93)
$$d_2^{ex} = \frac{ln\dfrac{V_0}{Nom} + \left(r-q-\dfrac{\sigma_V^2}{2}\right)T}{\sigma_V\cdot\sqrt{T}} = d_1^{ex} - \sigma_V\cdot\sqrt{T}\,.$$

We derive the value of the equity as a residual value as

(94)
$$E_0 = V_0 - D_0\,.$$

When allowing for different repayment agreements, the present value of the expected dividends is

(95)
$$
\begin{aligned}
VD_0 &= \sum_{t=1}^{T}\int_{t-1}^{t^-} E_0'(Div_\tau)\cdot e^{-r\cdot\tau}\,d\tau \\
&= \sum_{t=1}^{T}\int_{t-1}^{t^-} q\cdot E_0'(V_\tau \mid No\ Default\ until\ t-1)\cdot e^{-r\cdot\tau}\,d\tau
\end{aligned}
$$

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This can be simplified to

$$VD_0 = V_0 \cdot \left(e^q - 1\right)\sum_{t=1}^{T}\left[e^{-qt} \cdot N_{t-1}\left(d_1^{ex,1},\ldots,d_1^{ex,\,t-1};\rho_{t-1}\right)\right] \tag{96}$$

where

$$d_1^{ex,\tau} = \frac{\ln\left(\frac{V_0}{V_{T-\tau}^*}\right) + \left(r - q + \frac{\sigma_V^2}{2}\right)\cdot\tau}{\sigma_V \cdot \sqrt{\tau}} \tag{97}$$

and

$$d_2^{ex,\tau} = d_1^{ex,\tau} - \sigma_V \cdot \sqrt{\tau}. \tag{98}$$

We calculate the killing prices V_τ^* for each period recursively by ensuring that the boundary

$$E_{\tau^+}\left(V_\tau^*\right) = I_\tau + P_\tau \tag{99}$$

where

$$V_T^* = I_T + P_T \tag{100}$$

is fulfilled.

We determine the value of the firm ex-dividend, V_0^{ex}, as the difference between the value of the firm without dividend payments, V_0, and the present value of the expected dividends. We can write this as

$$\begin{aligned} V_0^{ex} &= V_0 - V_0\left(e^q - 1\right)\sum_{t=1}^{T}e^{-qt} \cdot N_{t-1}\left(d_1^{ex,1},\ldots,d_1^{ex,\,t-1};\rho_{t-1}\right) \\ &= V_0\left[1 - \left(1 - e^{-q}\right)\sum_{t=1}^{T}e^{-q(t-1)} \cdot N_{t-1}\left(d_1^{ex,1},\ldots,d_1^{ex,\,t-1};\rho_{t-1}\right)\right] \end{aligned} \tag{101}$$

The value of the firm's overall debt at $t=0$ is

$$\begin{aligned} D_0 &= V_0^{ex}\left[1 - N_T\left(d_1^{ex,1},\ldots,d_1^{ex,T};\rho_T\right)\right] + \sum_{t=1}^{T}\left(I_t + P_t\right)e^{-r\cdot t} \cdot N_t\left(d_2^{ex,1},\ldots,d_2^{ex,t};\rho_t\right) \\ &= V_0\left[1 - \left(1 - e^{-q}\right)\sum_{t=1}^{T} e^{-q(t-1)} \cdot N_{t-1}\left(d_1^{ex,1},\ldots,d_1^{ex,t-1};\rho_{t-1}\right)\right] \\ &\quad \cdot\left[1 - N_T\left(d_1^{ex,1},\ldots,d_1^{ex,T};\rho_T\right)\right] + \sum_{t=1}^{T}\left(I_t + P_t\right)e^{-r\cdot t} \\ &\quad \cdot N_t\left(d_2^{ex,1},\ldots,d_2^{ex,t};\rho_t\right). \end{aligned} \tag{102}$$

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2. Multiple Debt Instruments

Since we allow not only for different payback agreements but also for multiple debt instruments within one firm, we are also interested in determining the value of a specific debt instrument. For this, we once again use γ from equation (78) to express the relation between the specific debt instrument under consideration and the total debt. We can write the value of the specific debt instrument at $t=0$ when the firm pays continuous dividends to the equity holders as

$$
\begin{aligned}
D_0^S &= V_0^{ex}\left[\gamma_1 + \sum_{t=1}^{T-1}(\gamma_{t+1}-\gamma_t)\cdot N_t\left(d_1^{1,ex},\ldots,d_1^{t,ex};\rho_t\right) - \gamma_T\cdot N_T\left(d_1^{1,ex},\ldots,d_1^{T,ex};\rho_T\right)\right] \\
&\quad + \sum_{t=1}^{T}\left(I_t^S + P_t^S\right)e^{-r\cdot t}\cdot N_t\left(d_2^{1,ex},\ldots,d_2^{t,ex};\rho_t\right) \\
&= V_0\left[1-\left(1-e^{-q}\right)\sum_{t=1}^{T}e^{-q(t-1)}\cdot N_{t-1}\left(d_1^{ex,1},\ldots,d_1^{ex,\,t-1};\rho_{t-1}\right)\right]\cdot \\
&\quad \left[\gamma_1 + \sum_{t=1}^{T-1}(\gamma_{t+1}-\gamma_t)\cdot N_t\left(d_1^{1,ex},\ldots,d_1^{t,ex};\rho_t\right) - \gamma_T\cdot N_T\left(d_1^{1,ex},\ldots,d_1^{T,ex};\rho_T\right)\right]. \\
&\quad + \sum\nolimits_{t=1}^{T}\left(I_t^S + P_t^S\right)e^{-r\cdot t}\cdot N_t\left(d_2^{1,ex},\ldots,d_2^{t,ex};\rho_t\right).
\end{aligned} \tag{103}
$$

Of course, for the presented models with continuous dividends (102) and (103) the risk-neutral and the risk-averse recovery rates (45) and (59) have to be adjusted appropriately.

3. Numerical Example

We present a final numerical example illustrating the influence of dividends on the valuation of risky debt. For this, we again use the firm initially introduced in Section 2.6. The total assets amount to 100 and consist of a single debt instrument as well as equity. As in the first example, the debt instrument is a lump-sum loan with a nominal value of 70, annual interest payments of 2.5 %, and a maturity of five years. In contrast to the first example, however, the equity holders now receive continuous dividends. All other parameters remain unchanged (see Table 1).

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Table 5

Valuation Results for Firm with Single Lump-Sum Debt Instrument (Different Dividend Scenarios)

This table presents the valuation results for a firm with a single debt instrument and dividend-paying equity based on the parameters specified in Table 1. The debt instrument is a lump-sum loan with a nominal interest rate of 2.5 % p. a. The four numeric columns correspond to four valuation scenarios. In each scenario, the equity holders receive continuous dividends on the equity capital in the amount of q. The value of the risk-free debt, D_0^{rf}, is derived as the present value of the promised payments, while the value of the risky debt, D_0, is the present value of the expected payments.

	Continuous dividend payment rate q			
	0 %	1 %	2 %	3 %
Value of risk-free debt D_0^{rf}	70.58	70.58	70.58	70.58
Value of risky debt D_0	70.24	69.79	69.25	68.60

Table 5 provides an overview of how different continuous dividend payment rates influence the value of risky debt. The value of the risk-free debt at $t = 0$ is not influenced by the dividend payments since the calculation assumes that there is no risk of default. As can be seen from Table 5, the present value of the risky debt decreases with an increase in the continuous dividend payment rate. This is due to the fact that the dividends are paid out of the assets of the firm. The higher the dividend, the higher is the reduction of the assets and, in consequence, the lower is the value available to the debtholders in case of bankruptcy.

V. Concluding Remarks

In this paper, we contribute to the literature on the valuation of risky debt by providing three nested multivariate extensions of the standard Merton model. We follow a nested approach in the sense that each subsequent model extension contains the preceding model as a special case. Our extensions progressively relax some of the restrictive assumptions of the Merton model and, thus, provide a more realistic valuation framework than the standard model.

First, we lay forth an approach to pricing risky debt irrespective of the instrument's interest payment structure and principal repayment agreement. We provide repayment-specific closed-form solutions as well as a generic formula with which debt instrument with any kind of interest payment structure and repay-

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ment specification can be valued. We illustrate how the probability of default, the recovery rate, the distance to default, and the expected yield can be calculated for risky debt instruments from the point of view of both risk-neutral and risk-averse investors.

Second, we propose a technique for valuing multiple debt instruments issued by the same firm. We show that existing formulas for the valuation of debt should not be applied to single debt instruments if the debt instrument is part of a debt portfolio. This is the case because the default of one instrument in a debt portfolio has a direct impact on the other debt instruments in the portfolio. This interdependency diminishes the value of the other instruments and must, therefore, be incorporated into the valuation.

Third, we extend our generic formula for the valuation of single and multiple debt instruments irrespective of their indenture specifications to account for the effect of continuous dividend payments to the equity holders.

We complement each section with a numerical example in order to make the theoretical model more tangible and highlight the easy-of-use of our model for practical applications. We generally advise financial practitioners to use our adapted formulas when valuing debt or for the calculation of default and recovery rates. The standard Merton model, whose application is still widespread in practice, can lead to inaccurate results, thus compromising the informative value of a valuation. Based on the formulas presented here, the accuracy of the valuation results can be improved.

The model can be extended in several further dimensions. One potential avenue for future work could be the incorporation of a non-flat term structure as opposed to a constant risk-free interest rate. In this paper, we assume that the firm decides on the structure of its debt portfolio at $t=0$ and does not alter its composition until maturity. However, it can be meaningful for firms to evaluate the potential prepayment of existing debt instruments and refinance into new loans or bonds in order to take advantage of changes in the interest rate environment. The incorporation of such an evaluation into the models proposed here would require the departure from our assumption of a constant risk-free interest rate. Other models might instead incorporate a non-flat term structure to take such considerations into account. Another possible model extension is the inclusion of stochastic interest rates. In this paper, we limit the constituents of the firm's debt portfolio to fixed-interest debt instruments by assuming a constant deterministic interest rate. Other models may incorporate stochastic interest rates to enable the valuation of variable-rate loans and bonds. Finally, other studies could investigate the effects of subordination arrangements on our formulas for the valuation of debt.

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The debt in our consideration is defined as it usually exists in practice: Our debt is paying fixed coupons and possible principal repayments in discrete times and enables the existence of other debt instruments of the firm. Immediately before each payment date we endogenously determine an optimal killing price for the firm that may trigger bankruptcy. For the analysis of the debt's risk we calculate risk-neutral and risk-averse cumulative, total and conditional default probabilities as well as the recovery rates and the distances to default for each year until the debt's maturity. Regarding the yields to maturity our model not offers only the promised yield, but also the expected risk-neutral and the risk-avers yields. Therefore, in our opinion, our paper is an interesting contribution to the field of structural models for the valuation and detailed analysis of risky debt.

Appendix

The *Merton* (1974) model is the foundation for structural models used in the valuation and analysis of risky debt such as the model presented in this paper. For the sake of completeness, we therefore provide a brief review of the basics of the Merton model. For the mathematics of option pricing models see, e.g., *Hull* (2021).

In the Merton model, the value of the firm (i.e., the total assets), V_T, consists of two classes of claims: debt and equity. The debt of the firm consists of a single zero-coupon discount bond where the payment of the nominal value *Nom* is promised at maturity T. The firm's equity is seen as a residual claim. The total assets are distributed logarithmically normally and can be described by the diffusion-type stochastic process

(104) $$\frac{dV}{V} = \mu_V\, dt + \sigma_V\, dz$$

where μ_V refers to the drift, and σ_V is the volatility of the return on the firm per unit time, which is assumed to be constant. The risk-free interest rate r is also constant and there are no taxes, transaction costs, or dividends in the model. Investors are price takers, securities are freely divisible, and trading in the assets takes place continuously in time.

Under these assumptions, the firm's equity at maturity can be seen as a European call option written on the underlying asset V_T with exercise price *Nom* and maturity T such that

(105) $$E_T = \begin{Bmatrix} 0 \ if \ \ V_T < Nom \\ V_T - Nom \ if \ \ V_T \geq Nom \end{Bmatrix}.$$

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Since the value of the debt corresponds to the value of the total assets less the value of the equity, we have

(106) $$D_T = \begin{Bmatrix} V_T \ \ if \ \ V_T < Nom \\ Nom \ if \ \ V_T \geq Nom \end{Bmatrix}.$$

The firm's debt can be seen either as a long position in the firm's assets in combination with a short position in a call on the same or as a long position in a risk-free zero-coupon bond combined with a short put on the assets.

We can write the firm's equity and debt at time $t = 0$ as

(107) $$E_0 = V_0 \cdot N(d_1) - Nom \cdot e^{-r \cdot T} \cdot N(d_2)$$

and

(108) $$D_0 = V_0 \cdot [1 - N(d_1)] + Nom \cdot e^{-r \cdot T} \cdot N(d_2),$$

respectively, where

(109) $$d_1 = \frac{ln\frac{V_0}{Nom} + \left(r + \frac{\sigma_V^2}{2}\right)T}{\sigma_V \cdot \sqrt{T}},$$

(110) $$d_2 = \frac{ln\frac{V_0}{Nom} + \left(r - \frac{\sigma_V^2}{2}\right)T}{\sigma_V \cdot \sqrt{T}} = d_1 - \sigma_V \cdot \sqrt{T},$$

and $N(\cdot)$ denotes the standard normal cumulative distribution function.

We rearrange equation (108) to better illustrate the composition of the debt such that

(111) $$D_0 = Nom \cdot e^{-r \cdot T} - \left[Nom \cdot e^{-r \cdot T} - V_0 \cdot \frac{N(-d_1)}{N(-d_2)}\right] \cdot N(-d_2).$$

The first term on the righthand side of equation (11) corresponds to a long position in the risk-free bond while the second term is equal to the expected discounted loss. The latter is the product of the discounted loss given default (term in parenthesis) and the risk-neutral probability of default,

(112) $$PD = N(-d_2).$$

The probability of default is the probability that the firm will be unable to satisfy some or all of the requirements specified in the debt specifications (i.e., the bond indenture).

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One of the points of criticism against the Merton model and other structural models is that for the implementation in practice they require estimates for the value and the volatility of the firm's assets, which are not observable. But for companies with listed equity this problem can be solved (see *Merton* (1974))

(113) $$\sigma_E = \Delta_E \cdot \frac{V_0}{E_0} \cdot \sigma_V$$

where

(114) $$(\quad)$$

and the equity volatility σ_E is estimated empirically with the help of its historical or implied value. In this case V_0 and σ_V can be estimated by simultaneously solving (107) and (113) for the observable shareholder value E_0 and the estimated equity volatility σ_E. This approach can also be applied to our model with (52).

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