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Valuation at Origination of Legal Prepayment Options Embedded in 15-Year German Mortgage Loans

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Abstract

Section 489 of the German Civil Code anchors a prepayment option in all fixed-rate retail loans with a term of more than 10.5 years. The primary purpose of this paper is to develop an approach for valuing legal prepayment options (LPOs), embedded in 15-year German mortgage loans, at their origination. The analysis is based on 11,201 pairs of 10- and 15-year German mortgage rates that cover the period from June 2001 until February 2018 in steps of one month. In order to value the LPOs, trajectories of 10-year German mortgage rates are simulated by means of the exponential *Vasicek* model. The exercise strategy of the borrowers is a main driver of the value of LPOs. Our simulation results reveal that the following exercise strategy maximizes the average value of the LPOs under investigation (from the perspective of the day of their origination): On average, borrowers should exercise their LPOs, embedded in 15-year German mortgage loans, and refinance either if the present value of interest savings is at least 1.2 % of the outstanding loan amount or if the prevailing refinancing rate is, first, below the 15-year mortgage rate and, second, close to its presumed floor of 0.1 %.

Bewertung gesetzlicher Rückzahlungsoptionen in 15-jährigen Hypothekenkrediten am Tag ihrer Emission

Zusammenfassung

Nach Paragraph 489 des Bürgerlichen Gesetzbuchs haben Darlehnsnehmer das Recht, Darlehnsverträge mit gebundenem Sollzinssatz nach frühestens 10,5 Jahren ohne Vorfäl-

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Dedication: To my parents, Annette and Paul Manfred Wosnitza.

ligkeitsentschädigung vorzeitig zurückzuzahlen. Das Hauptziel des vorliegenden Aufsatzes besteht darin, ein Verfahren zur Bewertung von gesetzlichen Rückzahlungsoptionen, die 15-jährigen Hypothekenkrediten anhaften, am Tag ihrer Emission zu entwickeln. Die Analyse basiert auf 11.201 Paaren von 10- und 15-jährigen Hypothekenzinsen, die den Zeitraum von Juni 2001 bis Februar 2018 in monatlichen Schritten abdecken. Zur Bewertung des Rechts auf vorzeitige Rückzahlung werden 10-jährige Hypothekenzinsen mit Hilfe des exponentiellen *Vasicek* Models simuliert. Die Ausübungsstrategie der Kreditnehmer hat einen großen Einfluss auf den Wert der gesetzlichen Rückzahlungsoption. Unsere Simulationsergebnisse zeigen, dass die folgende Ausübungsstrategie den durchschnittlichen Wert der gesetzlichen Rückzahlungsoption (aus der Perspektive des Emissionstags) maximiert: Im Durchschnitt sind Darlehnsnehmer gut damit beraten ihren 15-jährigen Hypothekenkredit vorzeitig zu kündigen und zu refinanzieren, entweder wenn die Zinsersparnisse mindestens 1,2 % der noch ausstehenden Kreditsumme betragen oder wenn der Refinanzierungszins erstens unter dem 15-jährigen Hypothekenzins und zweitens in der Nähe seiner angenommenen Untergrenze von 0,1 % liegt.

Keywords: Constant default intensity; Exponential Vasicek model; Mortgage loan; Mortgage rate; Prepayment option; Refinancing.

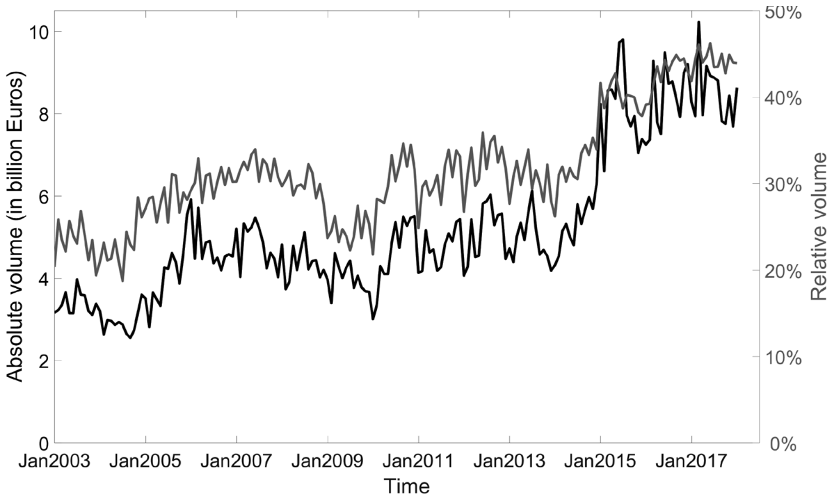
JEL classification: D14; G12; G21.

Abbreviations: Cumulative distribution function (CDF), Legal prepayment option (LPO), Probability of default (PD).

I. Introduction

The extremely loose monetary policy of the European Central Bank has brought interest rates, among others, on German mortgage loans into historical low territory. Under the assumption that interest rates are mean-reverting, a historically low interest rate environment implies that rates will rise in the future. By concluding long-term mortgage loans, borrowers can lock into low interest rates and, in doing so, protect themselves against raising rates. Following this logic, German private borrowers have increasingly concluded rather long-term mortgage loans during recent years. In particular, Figure 1 reveals that both the absolute volume (black line) and the relative volume (grey line) of new mortgage loans with an initial rate fixation of over 10 years, granted by German banks to private borrowers, has substantially increased in recent years. For example, the percentage of the volume of new mortgage loans with an initial rate fixation of over 10 years on the total volume of new mortgage loans has more than doubled during the period from January 2010 (21.8 %) until June 2017 (46.2 %).

Along with the growing volume of mortgage loans, whose rates are fixed over a period of more than 10 years, the volume of prepayment rights in the portfolios of German private borrowers has also increased. Section 489 (1) (2) of the German Civil Code entitles private borrowers to terminate a fixed-rate mortgage loan, in whole or in part, at any time observing a notice period of six



Source: The Underlying Time Series are Sourced From the Homepage of Deutsche Bundesbank.

Figure 1: Absolute (Left Axis and Black Line) and Relative (Right Axis and Grey Line) Volume of New Mortgage Loans with an Initial Rate Fixation of Over 10 Years Granted by German Banks to Private Clients Over the Time Period from January 2003 Until January 2018

months and a minimum waiting period of ten years from the day at which the loan was fully disbursed. In the event of termination, the borrower exchanges the unpaid balance on the mortgage loan for a release from further obligation; i. e. the borrower does not have to pay any prepayment penalty. The right to early terminate a mortgage loan without any prepayment penalty can be considered as a call option whose strike rate is equal to the contractually fixed mortgage rate. These legal prepayment options[1] (LPOs) give borrowers an incentive to refinance when the prevailing refinancing rate is lower than the contract rate on the existing mortgage loan, i. e. when interest rates have dropped (*Bennett* et al. 1999). Obviously, this financial incentive increases, ceteris paribus, with increasing interest rate differential.

Since banks act as counterparties to private borrowers, their short positions in prepayment options increased one-to-one with the volume of private borrowers' long positions over the last years. The increasing issuance of embedded LPOs, of course, intensified the need to value those, in particular, at their origination. Probably as a result of its high relevance for the banking sector, a rich body of literature on the valuation of embedded prepayment options has accumulated

[1] Hereinafter, prepayment rights pursuant to Section 489 (1) (2) of the German Civil Code are referred to as legal prepayment options (LPOs).

over the last decades (e.g. *Chen* et al. 2009; *Ding* et al. 2016; *Hilliard* et al. 1998; *Kalotay* et al. 2004; *Kau* et al. 1995). As the value of prepayment options depends very much on the exercise strategy of the borrowers, the strand of literature that deals with the valuation of prepayment options is strongly related to papers that develop and analyse exercise strategies for these embedded options (e.g. *Agarwal* et al. 2013; *Bennett* et al. 1999; *Bennett* et al. 2000; *Chen/Ling* 1989; *Kalotay* et al. 2008; *Stanton* 1995; *Virmani/Murphy* 2010).

The bulk of all these papers focuses on prepayment options embedded in US mortgage loans. *Cieleback* (2003) cites the higher number of securitizations as a possible reason for the dominance of US related research. In contrast to German mortgage loans, however, US mortgage loans can be prepaid without any prepayment penalty at any time. To the best of our knowledge, there is so far no literature that deals with the valuation of prepayment options embedded in German mortgage loans. Therefore, the primary purpose of this paper is to develop an approach for valuing LPOs, embedded in 15-year German mortgage loans, at their origination. As the exercise strategy of the borrowers is a main driver of the value of LPOs, the secondary purpose of this paper is to determine a simple and effective exercise strategy for the LPOs under investigation from the perspective of the day of their origination, i.e. under the condition that only mortgage rates until the origination of the LPO are known. In doing so, this paper is the first that accounts for the peculiarities of German mortgage loans, i.e. that fixed-rate mortgage loans can only be prepaid without any prepayment penalty after 10 years with a notice period of six months.[2]

The paper proceeds as follows. In Section 2, the theoretical approach for valuing LPOs, embedded in 15-year German mortgage loans, at their origination is developed. Section 3 presents three data sets. Based on these three data sets, the minimum option exercise threshold is determined in Section 4 by means of extensive Monte Carlo simulations. Section 5 concludes by framing a prepayment decision rule for 15-year German mortgage loans and by discussing the main limitations of this study.

[2] German bank mortgage loans are ideally suited for this investigation, because the terms and conditions that govern these loans are highly standardized. First, the interest rates are usually fixed over the entire life of the loans. Second, almost all German mortgage loans are structured as annuity loans. Under the terms of an annuity loan, the borrower periodically makes equal payments to the lender until maturity is reached. These instalments are typically paid monthly and are made up of an interest component and a principal component. As the outstanding balance of the loan diminishes over time, the interest rate component decreases while the principal component increases. Third, Section 489 (1) (2) of the German Civil Code anchors a prepayment option in all fixed-rate mortgage loans with terms of more than 10.5 years. This LPO entitles private customers to repay their fixed-rate mortgage loans, in whole or in part, at face value and without any redemption penalty at any time, observing a notice period of six months and a minimum waiting period of ten years from the date at which the mortgage loan was fully disbursed.

II. Theoretical Approach for Valuing LPOs, Embedded in 15-Year German Mortgage Loans, at Their Origination

The purpose of this section is to describe our theoretical approach for valuing LPOs, embedded in 15-year German mortgage loans, as a function of a predefined exercise threshold. More specifically, the approach determines the value of the LPOs at their origination. The value of an LPO is defined by the average present value of interest savings that a borrower can realise by exercising the LPO. Therefore, the exercise strategy of the borrower is a main driver of the value of the LPO. In this paper, it is assumed that private borrowers exercise their LPOs, embedded in 15-year mortgage loans, either

- if the present value of the potential interest savings is at least equal to a predefined exercise threshold or
- if the refinancing rate is lower than the rate on the 15-year mortgage loan and the present value of additional interest savings – under the condition that the refinancing rate drops to its presumed floor of 0.1 % – is less than EUR 250.

The first exercise trigger is based on a predefined threshold (i. e., a minimum present value of interest savings).[3] Throughout Section 2, this exercise threshold is considered as given. It is not expected that private borrowers define the exercise threshold by themselves. Rather, it is the objective of Section 4 to determine the exercise threshold that maximizes on average the value of the LPO. To this end, the average value of the LPO (i. e. the average present value of interest savings) is plotted against different exercise thresholds (which are expressed as a percentage of the outstanding loan amount at the time when the LPO is exercised) and, then, the exercise threshold leading to the highest average value of the LPO is selected.[4]

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[3] It is worth noting that the (minimum) exercise threshold corresponds to a minimum decrease in interest rates and, thus, also includes all exceeding downside movements of interest rates. For example, let us consider a 15-year mortgage loan with
- a fixed annual interest rate of 5 %,
- an annual amortization rate of 0 %,
- a time to maturity of 1.5 years (which implies a term of one year for the refinancing loan; see Assumption III on p. 7), and
- an outstanding loan amount of EUR 100,000.

Furthermore, let
- the minimum present value of interest savings be EUR 1,000,
- the discount factor be one, and let
- $\tilde{r}_{\text{ref}}$ denote the refinancing rate.

In this scenario, the potential interest savings, which are equal to $100{,}000 \cdot (5\% - \tilde{r}_{\text{ref}})$, have to be at least EUR 1,000 and, thus, the LPO is exercised if the refinancing rate is equal to or below $4\,\%$.

[4] Determining the optimal exercise threshold (i. e., finding the optimal trade-off between the frequency of option exercising and the amount of realised interest savings) is a

The rationale behind the second exercise trigger is as follows. In this paper, private borrowers assume that mortgage rates cannot fall below a floor of 0.1 %. Although the absolute interest rate differential is below the exercise threshold, it might still be beneficial for the private borrower to exercise the LPO if the prevailing refinancing rate is

- lower than the rate on the 15-year mortgage loan and
- only slightly above its presumed floor of 0.1 %,

because the impact of a further declining refinancing rate is very low in such a scenario. In order to account for this, this paper assumes that private borrowers exercise their LPOs also if, first, the refinancing rate is lower than the rate on the 15-year mortgage loan and, second, the private borrower can only realize additional interest savings below EUR 250 under the condition that the refinancing rate further drops to its presumed floor of 0.1 %.

The approach for valuing LPOs, embedded in 15-year German mortgage loans, at their origination requires as input simulated times of default of the private borrowers and simulated trajectories of potential refinancing rates. Before describing how these two kinds of input data are generated, this section proceeds with an overview of the main assumptions behind the valuation approach. At the end of this section, the core valuation approach is finally presented. In summary, the valuation approach for LPOs can be structured into five components which are addressed in the following subsections:

- Main assumptions,
- simulation of a private borrower's default time,
- simulation of a 10-year mortgage rate,
- calculation of a (potential) refinancing rate from a simulated 10-year mortgage rate, and
- valuation of LPOs embedded in 15-year mortgage rates.

non-trivial task. If the threshold is set too high, the spread between the rates on the existing and on the refinancing mortgage loan only widens sufficiently in rare cases. As a consequence, the prepayment option often expires unexercised. If the threshold is set too low, borrowers often exercise the prepayment option too early. In doing so, borrowers forgo the opportunity to realize additional interest savings by postponing refinancing until mortgage rates have declined even further (*Sharp* et al. 2009; *Chen*/*Ling* 1989). In summary, waiting to exercise has a cost (i. e. higher interest rates have to be paid for a longer period of time) as well as a potential benefit (i. e. the refinancing rate may decrease further).

1. *Main Assumptions*

The approach for valuing LPOs, embedded in 15-year German mortgage loans, at their origination is based on the following four fundamental assumptions. First, the possibility that borrowers prepay their mortgage loans from own savings is excluded. The reasoning behind this exclusion is the following. The monthly instalments of the mortgage loan reduce the disposable income of the borrower over the next years. Therefore, it appears unlikely that the borrower will accumulate sufficient savings to prepay the mortgage loan in 10.5 to 15 years. Under the assumption that the credit quality of the private borrower remains in the wide range to which the mortgage rates presented in Section 3 apply, by contrast, the private borrower has the opportunity to refinance his mortgage loan irrespective of his amount of available liquidity. Therefore, it appears to be reasonable to assume that exercising the LPO typically also involves taking out a new mortgage loan (Assumption I).

Second, there are two compelling arguments to assume that private borrowers prepay their mortgage loans in full rather than in part when exercising their LPOs. First, the absolute amount of money that a borrower saves by refinancing a 15-year mortgage loan is, ceteris paribus, an increasing function of the prepaid loan amount. Hence, the absolute savings of the borrower become maximal if the mortgage loan is prepaid in full at the optimal exercise time. Second, the administrative burden of private borrowers increases if they partially refinance their mortgage loans at different points in time (Assumption II).

It is a common practice among German banks to fix the rate on a retail mortgage loan up to six months in advance of the loan disbursement. Given this situation, the third fundamental assumption is that holders of a 15-year German mortgage loan simultaneously contract a new loan with a term of $15-(t_{ex}+NP)$ years (where NP stands for the notice period of six months), which is disbursed in six months, when exercising the LPO at time t_{ex}. In so doing, the private borrowers hedge against interest rate risk, i.e. they eliminate any uncertainty with regard to interest rate movements. Under this third assumption, the 15-year mortgage loan and the (potential) refinancing loan have the same maturity date (Assumption III).

Fourth, it is assumed that after private borrowers have exercised their LPOs, they have to borrow an amount that is equal to the principal outstanding on the terminated 15-year mortgage loan (Assumption IV). The last two assumptions ensure comparability between the original 15-year mortgage loan and the (potential) refinancing loan.

2. Simulation of Default Times of Private Borrowers

At the latest, a private borrower can exercise his LPO either six months before the mortgage loan expires or just before he defaults,[5] whichever occurs first. Thus, the latest possible exercise time is the minimum of the default time t_D and 15 years minus the notice period of six months. If the private borrower defaults before the planned exercise date, the LPO remains unexercised and, thus, its value becomes zero. Hence, the time of default of a private borrower is a driver of the value of the LPO. The objective of this subsection is to derive an equation, based on which default times of private borrowers can be simulated, in four steps.

In the first step, the main variables are introduced:

- Ω is the sample space,
- the random variable T_D stands for the time until default,
- t_D denotes a random realisation of T_D, and
- PD is the probability that the borrower defaults within a 1-year time period. Under the assumption of a constant default intensity, the probability of default in any 1-year time period is equal to PD.

Second, the probability that a default occurs until time t_N, i.e. $P[T_D \le t_N]$, is calculated. To this end, the time period from t_0 until t_N is divided into n equidistant intervals of length one year, i.e. $t_{k+1} - t_k = 1$ for all $k \in \{0,\ldots,N-1\}$, as illustrated in Figure 2.

Figure 2 reveals that the probability $P[T_D \le t_N]$ is equal to:[6]

$$P[T_D \le t_N] = \sum_{i=1}^{N} PD \cdot (1 - PD)^{t_i - t_0 - 1}. \tag{1}$$

As all n time intervals $t_{k+1} - t_k$ are of length one year, the difference $t_i - t_0$ can be replaced by i:

$$P[T_D \le t_N] = PD \cdot \sum_{i=1}^{N} (1 - PD)^{i-1}. \tag{2}$$

[5] For the sake of simplicity, it is assumed that a private borrower cannot anymore fall into default once he has exercised his LPO.

[6] An alternative starting point for arriving at equation (5) is the relationship between the probability to survive beyond time t_N, i.e. $P(T_D > t_N)$, and $P(T_D \le t_N)$:

$$\begin{aligned} P(T_D \le t_N) &= 1 - P(T_D > t_N) \\ &= 1 - (1 - PD)^{t_N - t_0} \\ &= 1 - exp[(t_N - t_0) \cdot \ln(1 - PD)]. \end{aligned}$$

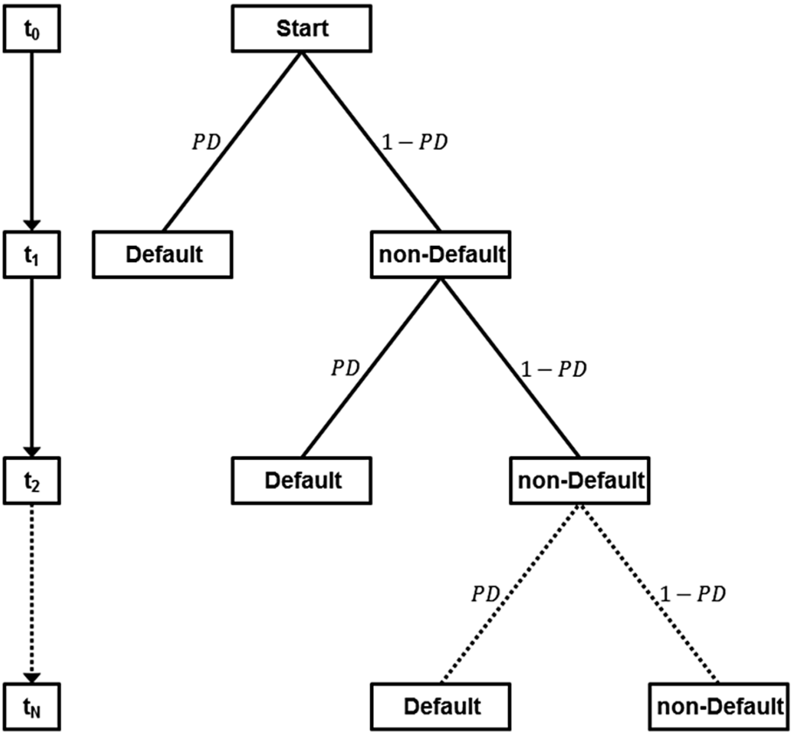

Figure 2: Probability of Default (PD) Tree

In the next step, the index is shifted, i.e. i is substituted by $j+1$:

$$P[T_D \leq t_N] = PD \cdot \sum_{j=0}^{N-1} (1-PD)^j . \tag{3}$$

The sum in equation (3) is the geometric series and, thus, this equation can be written as:

$$\begin{aligned} P[T_D \leq t_N] &= PD \cdot \sum_{j=0}^{N-1} (1-PD)^j \\ &= PD \cdot \frac{1-(1-PD)^N}{1-(1-PD)} \\ &= 1-(1-PD)^N \\ &= 1-exp[N \cdot \ln(1-PD)]. \end{aligned} \tag{4}$$

Replacing N with the difference $t_N - t_0$ yields:

$$P[T_D \leq t_N] = 1 - exp[(t_N - t_0) \cdot \ln(1 - PD)]. \tag{5}$$

Since t_D is a random realisation of the random variable T_D, the probability $P[T_D \leq t_D]$ is also a random variable. In the following, the probability $P[T_D \leq t_D]$ is also written as $F_{T_D}(t_D)$ where $F_{T_D}(\cdot)$ is the cumulative distribution function (CDF) of T_D. Based on this nomenclature, it is shown in the third step that $F_{T_D}(t_D)$ is uniformly distributed on the interval between 0 and 1. For this purpose, $\alpha \in [0;1]$ is introduced. The probability $P[F_{T_D}(t_D) \leq \alpha]$ is equal to:

$$\begin{aligned} P[F_{T_D}(t_D) \leq \alpha] &= P[\{\omega \in \Omega : F_{T_D}(T_D(\omega)) \leq \alpha\}] \\ &= P[\{\omega \in \Omega : T_D(\omega) \leq F_{T_D}{}^{-1}(\alpha)\}]. \end{aligned} \tag{6}$$

The CDF of T_D is $F_{T_D}(\cdot)$ and, thus, equation (6) transforms into:

$$\begin{aligned} P[F_{T_D}(t_D) \leq \alpha] &= F_{T_D}[F_{T_D}{}^{-1}(\alpha)] \\ &= \alpha \end{aligned} \tag{7}$$

As the probability $P[F_{T_D}(t_D) \leq \alpha]$ is equal to α, the random variable $F_{T_D}(t_D)$ is uniformly distributed on the interval from 0 to 1.

Fourth, the inverse function of $P[T_D \leq t_D] = F_{T_D}(t_D)$ is determined based on equation (4):

$$\begin{aligned} & F_{T_D}(t_D) = 1 - exp[(t_D - t_0) \cdot \ln(1 - PD)] \\ \Leftrightarrow\ & exp[(t_D - t_0) \cdot \ln(1 - PD)] = 1 - F_{T_D}(t_D) \\ \Leftrightarrow\ & (t_D - t_0) \cdot \ln(1 - PD) = \ln[1 - F_{T_D}(t_D)] \\ \Leftrightarrow\ & t_D - t_0 = \frac{\ln[1 - F_{T_D}(t_D)]}{\ln(1 - PD)}. \end{aligned} \tag{8}$$

If t_0 is equal to 0, equation (8) simplifies to:

$$t_D = \frac{\ln[1 - F_{T_D}(t_D)]}{\ln(1 - PD)}. \tag{9}$$

Hence, the time until default can be simulated by drawing a random number from the standard uniform distribution and plugging this random number into equation (9) for $F_{T_D}(t_D)$. The time until default t_D is an upper bound for the time of default. Thus, there is a certain degree of freedom in specifying the time of default. Here, we follow the convention and set the time of default equal to t_D.

3. *Simulation of Potential Refinancing Rates*

As neither the exercise time t_{ex} nor the future refinancing rate $r_{15-(t_{ex}+NP)}(t_{ex})$ (cf. Assumption III) is, of course, a priori known, the valuation of an LPO requires another simulation technique (in addition to the simulation of default times). Ideally, synthetic time series of potential future refinancing rates $r_{15-(t+NP)}(t)$ would be directly generated in order to value the LPO. As the data set, which is presented in Section 3, contains only rates of optionless 10-year mortgage loans, however, this approach is not feasible. Instead, the dynamics of the 10-year mortgage rate is modelled in a first step. Second, the refinancing rates, corresponding to the correct maturity date, are derived from the 10-year mortgage rate. The trajectory of potential refinancing rates then serves as input for valuing the LPO.

a) Choice of the Mortgage Rate Model

According to economic theory, the dynamics of interest rates are mean-reverting which means that interest rates are pulled back to a (relative or absolute) long-term equilibrium level over time (*van den End* 2013; *Hull* 2003). In fact, there are at least two compelling economic reasons in favour of the hypothesis that interest rates have some mean-reversion. First, high interest rates ceteris paribus increase the financing costs of industrial projects. This makes the projects unprofitable and, thus, they are either postponed or even completely cancelled. As a consequence, the economy tends to slow down and the demand for loans declines. The lower demand, in turn, results in declining interest rates. However, lower interest rates again stimulate the demand for loans as more projects become profitable (*Hull* 2003). Second, central banks use monetary policy measures in order to keep inflation (and thereby interest rates) within a target range defined by a lower and upper bound (*van den End* 2013).

The concept of mean reversion is included in many interest rate models, one of the earliest models was proposed by *Vasicek* (1977).[7] Elegant and simple as it is, the *Vasicek* model has a number of serious shortcomings. In particular, the short-rate may become negative with non-zero probability. Although the European Central Bank introduced a negative deposit facility interest rate in June 2014, this is still an unrealistic feature for mortgage rates. In order to avoid negative mortgage rates, the dynamics of the 10-year mortgage rate is modelled by the differential equation of the exponential *Vasicek* model:

[7] The *Vasicek* model is, for example, applied by *Zheng* et al. (2012) in order to determine the optimal refinancing strategy for mortgage borrowers in a stochastic interest rate environment.

(10) $$d\ln\left[r_{10\,yr}(t)\right] = \left\{\tilde{\alpha} - \tilde{\beta}\cdot\ln\left[r_{10\,yr}(t)\right]\right\}\cdot dt + \tilde{\sigma}\cdot dW_t$$

where

- $r_{10\,yr}(t)$ is the rate of a 10-year mortgage loan at time t,
- $\tilde{\alpha}$ denotes the long-term mean reversion level, i.e. all future trajectories of $\ln\left[r_{10\,yr}(t)\right]$ will evolve around $\tilde{\alpha}$ in the long run,
- $\tilde{\beta}$ is related to the speed of mean reversion,
- dt is an infinitisimal time step,
- $\tilde{\sigma}$ represents the diffusion parameter that controls the amplitude of randomness entering the system instant by instant, and
- dW_t is an infinitesimal increment of a Wiener process modelling the continuous inflow of randomness into the system.

The stochastic process in equation (10) is completely defined by the three time-independent parameters $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\sigma}$ together with the initial condition $r_{10\,yr}(0)$. The term $\left\{\tilde{\alpha} - \tilde{\beta}\cdot\ln\left[r_{10\,yr}(t)\right]\right\}$ in equation (10) is the drift factor. It represents the expected instantaneous change in interest rate at time t. The drift factor pulls the logarithm of the 10-year mortgage rate towards its long-term mean reversion level $\tilde{\alpha}$ with magnitude proportional to the deviation of $\tilde{\beta}\cdot\ln\left[r_{10\,yr}(t)\right]$ from $\tilde{\alpha}$. Just like the *Vasicek* model, the exponential *Vasicek* model thus incorporates mean-reversion. This deterministic drift is superimposed by the normally distributed stochastic term $\tilde{\sigma}\cdot dW_t$.

b) Calibration of the Mortgage Rate Model

After selecting the interest rate model, it has to be calibrated to real world data. To this end, equation (10) is first discretised:[8]

(11) $$\ln\left[r_{10\,yr}(t_{k+1})\right] - \ln\left[r_{10\,yr}(t_k)\right] = \left[\alpha - \beta\cdot\ln\left[r_{10\,yr}(t_k)\right]\right]\cdot\Delta t + \sigma\cdot\varepsilon$$

where

- t_k is the k-th time step,
- Δt is the predefined difference between two consecutive time steps, i.e. $\Delta t = t_{k+1} - t_k$, and
- ε denotes a normally distributed random variable with mean equal to zero and standard deviation equal to one.

[8] In Section 4, a discretisation in steps of one month is chosen which means $\Delta \mathrm{t} = 1$ month.

Rearranging equation (11) yields:

$$\ln\left[r_{10\,yr}\left(t_{k+1}\right)\right] = \left[1 - \beta \cdot \Delta t\right] \cdot \ln\left[r_{10\,yr}\left(t_k\right)\right] + \alpha \cdot \Delta t + \sigma \cdot \varepsilon. \tag{12}$$

Before equation (12) can be used to generate synthetic time series of $r_{10\,yr}(t)$, the parameters α, β, and σ have to be determined. As no prepayment options on mortgage loans are traded on the market, the three parameters cannot be derived from market prices. However, equation (12) reveals that the three parameters can be calibrated by means of linear regression between the dependent variable $\ln\left[r_{10\,yr}\left(t_{k+1}\right)\right]$ and the explanatory variable $\ln\left[r_{10\,yr}\left(t_k\right)\right]$. More precisely, the slope of the linear function is equal to $\left(1 - \beta \cdot \Delta t\right)$ and the interception is equal to $\alpha \cdot \Delta t$. As Δt is a predefined parameter, the parameters α and β follow from these two relationships. Furthermore, it follows from equation (12) that the variance of the residuals $\ln\left[r_{10\,yr}\left(t_{k+1}\right)\right] - \left(\left[1 - \beta \cdot \Delta t\right] \cdot \ln\left[r_{10\,yr}\left(t_k\right)\right] + \alpha \cdot \Delta t\right)$ is equal to the variance of $\sigma \cdot \varepsilon$. As ε is a standard normally distributed random variable, the variance of $\sigma \cdot \varepsilon$ is equal to σ^2. Thus, the parameter σ is equal to the standard deviation of the residuals. After calibration, equation (12) can be used in order to simulate time series of 10-year mortgage rates.

c) Derivation of the Potential Refinancing Rates

As already pointed out, the valuation of the LPO does not require the future 10-year mortgage rate, but the future refinancing rate $r_{15-(t_{ex}+NP)}\left(t_{ex}\right)$ (cf. Assumption III). In order to calculate potential refinancing rates from a simulated 10-year mortgage rate, two assumptions are made. First, it is assumed that the term structure of mortgage rates at any point in time can be derived by parallel shifting the Pfandbrief curve by the difference between the 10-year mortgage rate and the 10-year Pfandbrief rate at $t = 0$. More concretely, the -year tenor point of the mortgage curve at $\tilde{t}$ is then equal to the m-year tenor point of the Pfandbrief curve at $\tilde{t}$ plus the difference between the 10-year mortgage rate and the 10-year Pfandbrief rate at $t = 0$. The assumption that the mortgage curve can be derived from the Pfandbrief curve by adding a constant term (i.e. an interest margin for the bank) seems to be reasonable, because banks often issue Pfandbriefe in order to finance mortgage loans and, thus, Pfandbrief rates should be highly correlated with mortgage rates. However, this approach could, at least in principle, lead to negative (potential) refinancing rates. Therefore, the (potential) refinancing rates are floored at 0.1 %. Second, it is assumed that the term structure of mortgage rates retains its shape over the course of time.[9] Given these two assumptions, plus a simulated trajectory of monthly 10-year mort-

[9] Please note that this feature of the interest rate model is identified and discussed as a limitation in the Conclusion.

gage rates over a period of 14.5 years, the potential refinancing rates $r_{15-(t+NP)}(t)$ can be calculated for $t \in [10;\min(14.5;t_D)]$.

4. *Valuation at Origination of an LPO Embedded in a 15-Year German Mortgage Loan*

So far, it was shown how the times of default of private borrowers and the trajectories of (potential) refinancing rates are simulated. The purpose of this subsection is to explain how LPOs, embedded in 15-year German mortgage loans, are valued as a function of a predefined exercise threshold based on these input variables.

In order to value the LPO (i.e. to determine the present value of interest savings) for a given exercise threshold, the synthetic trajectory of potential refinancing rates[10] $r_{15-(t+NP)}(t)$ is considered in the interval $t \in [10;\min(14.5;t_D)]$. Starting at $t = 10$, the time series is analysed month by month either until the time variable t reaches the minimum of t_D and 14.5 years or until, at least, one of the two LPO exercise conditions, outlined on p. 469, is fulfilled, whichever occurs first. In order to check whether an LPO exercise condition is fulfilled, a mortgage loan with

- interest rate equal to $r_{15-(t+NP)}(t)$,
- maturity in $15-(t+NP)$ years (cf. Assumption III),
- loan amount equal to the balance of the 15-year mortgage loan at $t + NP$ (cf. Assumption IV), and
- monthly payments equal to those of the 15-year mortgage loan

is structured at each point in time t. Due to the last of the above bullet points, the cash-flow profiles of the 15-year mortgage loan and the potential refinancing loan are the same except for the residual loan amount at $t = 15$ years. Thus, the nominal amount of potential interest savings for the private borrower is equal to the difference between the residual amounts at maturity of the 15-year mortgage loan and the potential refinancing loan. In order to check whether the LPO is exercised, the potential interest savings are discounted to time t. Either if the present value of interest savings is at least equal to the predefined exercise threshold or if the prevailing refinancing rate is, first, below the 15-year mortgage rate and, second, close to its presumed floor of 0.1 %, it is assumed that the LPO is exercised. In this case, the value of the LPO is given by the discounted interest savings. Otherwise, the procedure is repeated for the next point in time.

[10] According to Assumption III, the term of the (potential) refinancing loan is equal to the residual term of the 15-year mortgage loan minus the notice period of six months.

So far, a theoretical approach for valuing an LPO, embedded in a 15-year German mortgage loan, as a function of a predefined exercise threshold has been described. This valuation method is based on a single trajectory of potential refinancing rates and on a single time of default of a private borrower. As interest rates follow stochastic processes and the time of default of the borrower is also subject to uncertainty, however, the valuation approach has to be repeated for many trajectories of refinancing rates and many realisations of default times. Then, the value of the LPO for a predefined exercise threshold is given by the average value of the LPO over all these simulated scenarios. As outlined in the beginning of Section 2, the objective of this paper is to determine the value of the LPO as a function of the exercise threshold. Therefore, this valuation approach is repeated for many different exercise thresholds in Section 4.

III. Data

This section presents the three data sets on which the empirical analysis, presented in Section 4, is based. First, Pfandbrief rates are needed. As outlined on p. 478 the potential interest savings are discounted to the point in time at which the LPO could be exercised. The required discount rates are derived from the German Pfandbrief curve. The Pfandbrief time series are sourced from the homepage of Deutsche Bundesbank.

Second, the probability that a borrower defaults during any 1-year period, i.e. PD, is required in order to calibrate equation (9). For this purpose, observed annual default rates of residential mortgage loans over the time period from 2006 until 2015 are collected from a bank that holds a large German mortgage portfolio. From this time series, a conservative through-the-cycle estimate for the annualized default probability of private borrowers is determined.

Third, the analysis is based on a data set that consists of monthly pairs of 10- and 15-year German mortgage rates. To be precise, only the mortgage rates of banks, for which both a 10- and a 15-year mortgage rate are available, are included in this data set. The number of banks that contribute to the data set varies from month to month. On overage, pairs from 56.29 banks are available per month with the minimum number of banks being 39 and the maximum number being 99. In total, the data set consists of 11,201 pairs of 10- and 15-year mortgage rates that apply to mortgage loans with loan-to-value ratios not higher than 90%. As the latest 10-year time window of 10-year mortgage rates is used to calibrate the interest rate path generator for each month, the average present value of the LPO is calculated from 5,114 (covering the time period from June 2011 to February 2018) of the total 11,201 pairs.

The set of mortgage rates was hand-collected from a German financial magazine and it covers the period from June 2001 until February 2018.[11] This is a representative period that includes:

- the 11th September 2001 terrorist attacks and the following recession accompanied by an aggressive cut in interest rates (2001–2003),
- a period of normal to high growth paralleled by a gradual increase in interest rates (2004–2007),
- a severe financial crisis associated with another aggressive reduction of interest rates (2008–2009),
- a low growth period with high macroeconomic uncertainty and historically low interest rate environment (2010–2012), and again
- a period of normal to high growth in a historically low interest rate environment (2013–2018).

The following two tables show the univariate descriptive statistics of the analysed 5,114 pairs and of the total 11,201 pairs of 10- and 15-year mortgage rates, respectively.

Table 1

Univariate Descriptive Statistics of the Analysed 5,114 Pairs of 10- and 15-Year Mortgage Rates (Author's Own Calculation)

	10-year mortgage rate	*15-year mortgage rate*
Standard deviation	0.82 %	0.84 %
Average	1.92 %	2.39 %
Minimum	0.70 %	1.07 %
1 %-quantile	0.81 %	1.20 %
5 %-quantile	0.97 %	1.38 %
25 %-quantile	1.26 %	1.70 %
50 %-quantile	1.60 %	2.08 %
75 %-quantile	2.53 %	3.06 %
95 %-quantile	3.45 %	3.96 %
99 %-quantile	4.08 %	4.57 %
Maximum	4.48 %	5.13 %

[11] Please note that no mortgage rates are available for September 2002 and May 2003.

Table 2

Univariate Descriptive Statistics of the Analysed 11,201 Pairs of 10- and 15-Year Mortgage Rates (Author's Own Calculation)

	10-year mortgage rate	*15-year mortgage rate*
Standard deviation	1.53 %	1.47 %
Average	3.38 %	3.78 %
Minimum	0.70 %	1.07 %
1 %-quantile	0.85 %	1.28 %
5 %-quantile	1.07 %	1.50 %
25 %-quantile	1.73 %	2.23 %
50 %-quantile	3.79 %	4.18 %
75 %-quantile	4.65 %	4.99 %
95 %-quantile	5.53 %	5.82 %
99 %-quantile	6.08 %	6.43 %
Maximum	6.54 %	6.94 %

IV. Valuation at Origination of LPOs Embedded in 15-Year German Mortgage Loans

Based on the assumption that holders of a 15-year mortgage loan exercise their LPOs either

- if the present value of the potential interest savings is at least equal to a predefined exercise threshold or
- if the refinancing rate is lower than the rate on the 15-year mortgage loan and the present value of additional interest savings – under the condition that the refinancing rate drops to its presumed floor of 0.1 % – is less than EUR 250,

the average present value of the interest savings (i.e. the average value of the LPO) as a function of the exercise threshold is calculated from the 5,114 pairs of 10- and 15-year mortgage rates in the following seven steps.

(1) Equation (9) is calibrated simply by inserting the annual probability of default parameter PD.

(2) An exercise threshold is defined. The average present value of interest savings (i.e. the value of the LPO) is determined for exercise threshold values in the range from 0 % to 10 % of the outstanding loan amount at the time when the LPO is exercised.

(3) For every month, the interest rate path generator in equation (12) is calibrated based on the latest 10-year time window of average 10-year mortgage rates. As a rolling time window of average 10-year mortgage rates is used as

calibration data base, only the oldest average 10-year mortgage rate is replaced by a new average 10-year mortgage rate for each month. Hence, the calibration data base only changes marginally from one month to the next.

On the one hand, a 10-year time period should approximately cover a full cycle of the German economy. On the other hand, calibrating the interest rate path generator based on a longer period implies that the calibration is done on older and, thus, likely less representative data. Therefore, the period length of 10 years is deemed appropriate.

Technically, this second kind of calibration is done by means of linear regression. If the interest rate path generator in equation (12) was calibrated on a time series that has a trend, it would (on average) generate synthetic interest rate trajectories with a similar trend. As it is assumed in this paper that the historical trend of the average 10-year mortgage rate is not necessarily representative for its future development, a detrended data set is used for the calibration. In order to obtain a detrended calibration data set, the 10-year time window of the average 10-year mortgage rates is flipped and, then, the original and the flipped 10-year windows are merged into the calibration data set before performing the linear regression.

The following table shows the minimum, median, average, and maximum R^2-values over all 81 linear regressions (from June 2011 to February 2018). As all R^2-values are well above 90 %, the calibration results are deemed appropriate. The minimum, median, average, and maximum values of the estimated coefficients α, β, and σ from equation (12) are also provided in Table 3. As the calibration data basis only changes marginally from one month to the next, the estimates of the coefficients also remain relatively constant in the short term. Over the entire period from June 2011 to February 2018, however, the estimated coefficient values change substantially as the relatively high differences between the minimum and maximum values for α, β, and σ show, respectively.

Table 3

Summary of Linear Regression Results (Author's Own Calculation)

	α	β	σ	R^2
Minimum	–1.497	0.069	0.035	0.921
Median	–0.541	0.165	0.040	0.973
Average	–0.664	0.206	0.043	0.966
Maximum	–0.250	0.483	0.054	0.988

In addition, the null hypothesis that the residuals are normally distributed is tested by means of the Anderson Darling test and the Kolmogorov Smirnov test. The minimum and maximum p-values of the Anderson Darling test (resp. of the Kolmogorov Smirnov test) over the 81 linear regressions are 19.91 % and 93.52 % (resp. 37.89 % and 96.56 %). Hence, the null hypothesis is not rejected on a two-sided confidence level of 5 % and, thus, minimizing the sum of squared residuals results very likely in the best linear unbiased estimator.

(4) For each of the predefined exercise thresholds, 1,000 paths of the (average) 10-year mortgage rate, each covering 14.5 years, are simulated in one month time steps. Starting point of the interest rate simulation is the current 10-year mortgage rate of each bank. These simulated interest rate trajectories serve as input for the valuation approach outlined in Subsection 2.4.

(5) A time of default is simulated for each interest rate path by means of equation (9). As borrowers cannot exercise the LPO after default, the interest rate path is stopped at the minimum of t_D and 14.5 years as explained in Subsection 2.2.

(6) Following the procedure described in Subsection 2.4, the nominal value of interest savings (i. e. the undiscounted value of the LPO) is determined and discounted to the point in time at which the option is exercised in each of the 1,000 scenarios.

(7) The value of the LPO is calculated as the average present value of interest savings over all 1,000 simulated interest rate trajectories (c.f. Subsection 2.4).

For each threshold value defined in step (2) and for each pair of 10- and 15-year mortgage rates between June 2011[12] and February 2018, steps (3) to (7) are repeated and the average value of the LPO over the 5,114 pairs of mortgage rates is determined, respectively. These average values of the LPO are plotted against the respective exercise thresholds in Figure 3 (and Figure 4). The average value of the LPO as a function of the exercise threshold (expressed as a percentage of the outstanding loan amount at the time when the LPO is exercised) has a global maximum value at 1.2 %. In other words, the exercise threshold of 1.2 % leads to a maximum average value of the LPO of EUR 2073.86 for an initial loan amount of EUR 100,000. Based on this result, a decision rule (from the perspective of the day of the LPO's origination) is framed in the Conclusion. Figure 4 shows the results around the maximum value in more detail. The underlying data of Figure 4 reveal that at least 99 % (resp. 97 %) [resp. 95 %] of the maxi-

[12] As a rolling time window of the latest 10 years of 10-year mortgage rates is used in order to calibrate equation (4) for each month, the first present values of interest savings are calculated for June 2011.

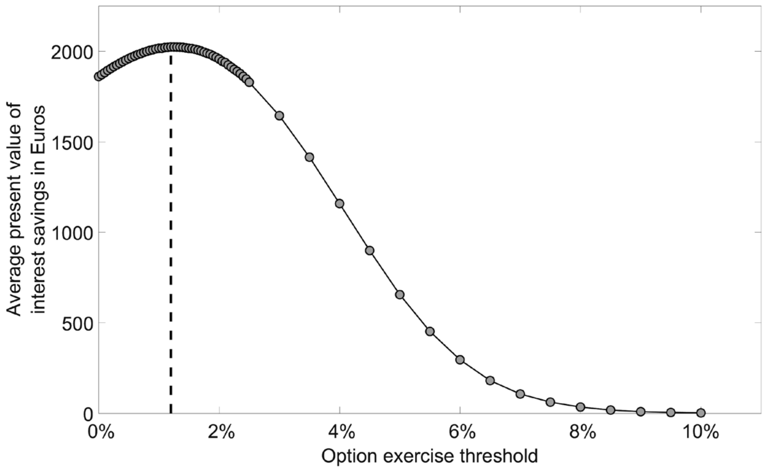

Figure 3: The Average Present Value of Interest Savings Over the 1,000 Interest Rate Scenarios (i.e. the Value of the LPO) and Over the 5,114 Pairs of Mortgage Rates as a Function of the Exercise Threshold in the Range from 0% to 10%. The Scattered Vertical Line Indicates the Optimal Exercise Threshold at 1.2%

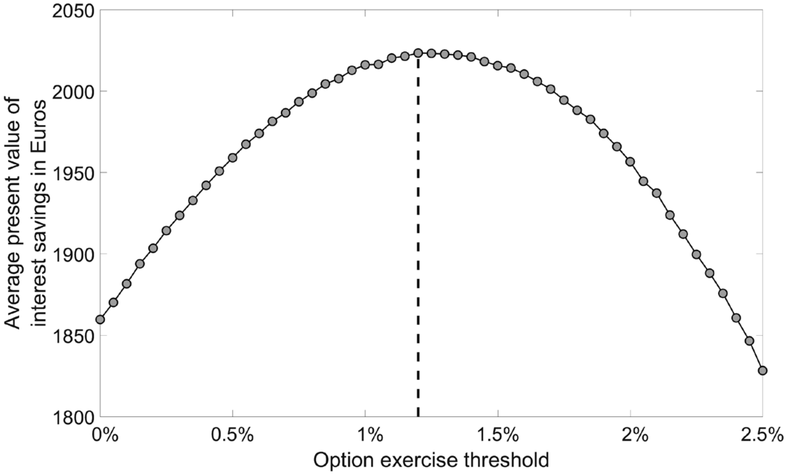

Figure 4: The Average Present Value of Interest Savings Over the 1,000 Interest Rate Scenarios (i.e. the Value of the LPO) and Over the 5,114 Pairs of Mortgage Rates as a Function of the Exercise Threshold in the Range from 0% to 2.5%. The Scattered Vertical Line Indicates the Optimal Exercise Threshold at 1.2%.

mum interest savings are realized if the exercise threshold falls in the interval between 0.85 % and 1.65 % (resp. 0.55 % and 1.95 %) [resp. 0.3 % and 2.15 %] of the outstanding loan amount at the time when the LPO is exercised.

V. Conclusion

According to Section 489 (1) (2) of the German Civil Code, German banks are legally obliged to accept full or partial prepayments of fixed-rate mortgage loans without any prepayment penalty after 10.5 years of the complete loan disbursement. This statutory prepayment right can change the flow of wealth between households and banks. By early terminating their mortgage loans and refinancing them at lower rates, borrowers can realize substantial interest savings. Offsetting these benefits are the costs of the mortgage lenders. When a large number of borrowers refinances mortgage loans with high interest carries in a low interest rate environment, banks' interest incomes might be reduced (*Bennett* et al. 1999). Hence, it is an economic necessity for banks to value LPOs at their origination. This paper develops a valuation approach for LPOs, embedded in 15-year mortgage loans, that realistically accounts for the amortization characteristics and the default risk[13] of private borrowers.

The exercise strategy of the private borrowers drives the value of LPOs to a significant extent. In this paper, it is assumed that private borrowers follow a simple exercise strategy which they define at the origination of the LPO and do not change afterwards.[14] The simulation results reveal that the following decision rule maximizes the average value of the LPO over the 5,114 pairs of mortgage rates: On average, borrowers should exercise their LPOs, embedded in 15-year German mortgage loans, and refinance either if the present value of interest savings is at least 1.2 % of the outstanding loan amount or if the prevailing refinancing rate is, first, below the 15-year mortgage rate and, second, close to its presumed floor of 0.1 %. This exercise strategy leads to a maximum average value of the LPO of EUR 2073.86 for an initial loan amount of EUR 100,000.

In addition to banks, these results are also relevant for private borrowers in, at least, two respects. First, the presented exercise strategy could help private borrowers to mitigate their behavioural biases that lead to a suboptimal usage of LPOs and, thus, to exercise the LPOs with more financial perfection. It is worth noting, however, that the exercise strategy maximizes the average value of the LPO only from the perspective of the day of its origination. If private borrowers take the latest level of interest rates into account when deciding whether or not

[13] For the sake of simplicity, however, it is assumed that a private borrower cannot anymore fall into default once he has exercised his LPO.

[14] The assumption of an unchanging exercise strategy is necessary in order to determine the value of the LPO at its origination.

to exercise the LPO, the value of the LPO could very likely be increased even further. Moreover, the prepayment strategy is optimized based on average LPO-values over 5,114 pairs of mortgage rates. For individuals, other exercise thresholds could result in higher interest savings. Despite these two points of criticism, this is the first paper that provides German private borrowers with guidance on the question of when to prepay a 15-year mortgage loan. Second, 15-year German mortgage loans can be viewed as ordinary (i. e. non-prepayable) debt instruments with a prepayment option attached to them. Hence, the presented approach for valuing LPOs at their origination allows private borrowers, at least in principle, to calculate whether an optionless 10-year or a 15-year mortgage loan is cheaper.

Not all aspects of the real world can be modelled and each model therefore simplifies the reality by making assumptions which can, in turn, lead to limitations. Specifically, the reader should remain aware of the following three fundamental assumptions of this study when evaluating the practical relevance of this paper. First, this paper assumes that private borrowers exercise their LPOs, embedded in 15-year German mortgage loans, with financial perfection. Although a rich body of literature suggests that the interest rate differentials are indeed the most important driver of prepayment decisions of private borrowers (e. g. *Beckett*i 1989; *Kau/Springer* 1992; *Kolbe/Zagst* 2008; *Perry* et al. 2001), another strand of literature provides evidence that private borrowers do not exercise their prepayment options as ruthlessly as option theory suggests (e. g. *Archer/Ling* 1993; *Chen/Ling* 1989; *Kau* et al. 1992; *Quigley/Van Order* 1990; *Schwartz/Torous* 1989; *Sharp* et al. 2009).[15] In contrast to other prepayments (also referred to as turnover), interest rate-induced prepayments benefit private borrowers strictly at the expense of banks (*Kalotay* et al. 2004). Therefore, the value of prepayment options should indeed be mainly attributed to interest-rate induced prepayments rather than to turnover as these are detrimental to banks'

[15] For example, *Agarwal* et al. (2016) find that approximately 57 % of the borrowers refinance suboptimally. However, the fact that borrowers do not exercise their prepayment options with such a financial perfection as the theory of options suggests does not necessarily indicate that their prepayment behaviour is irrational. In addition to the prepayment option, the borrower owns other financial assets such as the underlying real estate property (*Archer/Ling* 1993; *Hall* 1985). As the values of the real estate property and the prepayment option are not perfectly correlated, the strategy that maximizes the return on the real estate property is not necessarily identical with the strategy that maximizes the return on the prepayment option (*Archer/Ling* 1993; *Hall* 1985). Thus, the private customer cannot simultaneously maximize the returns from both assets. Since private customers would behave rationally if they maximize their total wealth, the prepayment option will not necessarily be exercised when the value of the option is maximal (*Hall* 1985). There is indeed empirical evidence that borrowers in locations where house prices rose significantly are much more prone to early repay their mortgage loans than borrowers whose house prices increased less strongly in value (*Mayer* et al. 2013).

interest incomes (*Kalotay* et al. 2004). However, assuming that a higher percentage of private borrowers exercises their LPOs with financial perfection than actually expected leads to an overvaluation of the LPO.

Second, it is a common practice of German banks to accept mortgage prepayments of 5 % of the initial loan amount per year without any prepayment penalty. However, voluntarily granted partial prepayment rights are neglected within this paper. The reason for this is as follows. Due to the relatively low amount of partial prepayments, private borrowers cannot conclude a new mortgage loan in order to prepay. Instead, they have to use, for example, their own savings. As no information on the amount of savings of individual private borrowers is publically available, it is assumed that these additional partial prepayment rights are not used at all. Neglecting partial prepayment rights results in an overestimation of future residual mortgage loan amounts, which in turn transpires into an overestimation of the present value of the interest savings.

As a consequence of the first two assumptions, the valuation approach results in a higher value of the LPO than expected from the perspective of the day of its origination. Therefore, banks can use this approach in order to conservatively price LPOs embedded in 15-year mortgage loans.

Third, a simple 1-factor interest rate path generator, which is based on the exponential *Vasicek* model, is applied.[16] Although the applied interest rate path generator cannot model shifts in the term structure that are different at different tenor points, the model seems to be appropriate for modelling the overall level of the term structure of mortgage rates. According to a principal component analysis performed by Credit Suisse (2012), the level of yields explains most of the dispersion of term structures of interest rates and, thus, it is the most important factor for adequately modelling term structures of interest rates. Therefore, the applied 1-factor interest rate path generator is deemed appropriate.

In view of these three assumptions or limitations, the simulation results must be viewed with a certain degree of caution by mortgagors, advisors, and bankers.

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